SOME NOTES ON THE ORDER OF MAGNITUDE OF EFFECTS INVOLVED WITH THE PARAMETERISATION OF TOPOGRAPHY IN NUMERICAL MODELS

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1. Introduction

Whilst there is no doubt that all scales of topography exert a significant influence on atmospheric flow little quantitative information exists, and even our qualitative understanding is incomplete. Our weaknesses are both theoretical and observational.

This document seeks to provide a simple description of the processes involved in momentum transfer between the Earth and its atmosphere.

2. Form drags

We may define form drag as being due to the viscous dissipation of the standing disturbance produced by the topography. This definition admits atmospheric effects on two widely different scales. On short scales it covers the "aerodynamic" forces arising in flow over obstacles. Whilst on the large atmospheric scales it includes the dissipation, through the boundary layer pumping mechanism, of the relative vorticity produced by the compression of fluid filaments passing over the topography.

a. Small scale "aerodynamic" forces

On the scales of vegetation, trees and buildings these forces are considered part of the turbulent atmospheric boundary layer. The Reynolds numbers of these scales are large and these effects can be parameterised through a drag coefficient. For example the surface stress $\tau = \rho C_G U_g^2$ where U_g is the geostrophic wind and C_G the so called "geostrophic" drag coefficient. Having been subject to extensive laboratory and field measurements we have some ideas on the relationship of C_G to the nature of the terrain. This usually involved a discussion of the "roughness length" of the terrain. For a smooth surface such as the sea in light winds there is no topography and the stress is communicated by viscous forces alone. C_G has a value of about $0.5 . 10^{-3}$. However as we proceed to, say, short grass and to trees C_g might have values of $\sim 1.0 . 10^{-3}$ and $\sim 2.0 . 10^{-3}$ respectively. Thus on these scales C_G is not a strong function of the terrain.

If we consider a single element of topography with height h and scale L exposed to a flow U_o the "aerodynamic" pressure force will, if the only force, give a stress

$$\sim \frac{1}{2} C\rho U_o^2 \, {}^h/L \qquad \text{where C is drag coefficient}$$

which will be O(1) for bluff obstacles but less for smooth bodies. Such a pressure force is clearly unable to give a direct account of observed effects on small scales. The situation is complex due to the turbulent nature of the flow, the influence of one element on another, and the surface mounting of the elements.

We have very little information on the effect of larger scale, say 100 m to 10^4 m, topography. If the pressure forces given above were to be realised as a net force, the effects of topography on this scale would be dramatic. On these scales there is almost no reliable data but indirect inferences based on aircraft observations suggest that extrapolation from smaller scales is the current best estimate. In this case even mountains might only have $C_G \sim 3.10^{-3}$. The actual effects on this scale could be very different but as yet there are no sound sources of data, or theories.

As a first step in this direction Mason and Sykes (1978, 1979) have made numerical integrations of the Navier-Stokes equations to examine laminar flows over surface mounted obstacles. The results serve to illustrate possible pitfalls but have no direct application to turbulent flows.

For 2-D flows they found that for steady flows the pressure force was nearly balanced by a reduction in viscous stress giving a very small net effect. However at high Reynolds numbers and steep slopes unsteady flow occurred with eddy shedding and here large drags of the order of the pressure force were found. For 3-D flows at low Reynolds numbers and/or shallow slopes little net effect was found,as with the steady 2-D case,but for steep higher Reynolds number cases, even though the flow was steady, a large force of the order of the pressure force resulted. In all these later cases the flow had separated upstream and generated "horse shoe" eddies. These longitudinal eddies arise as the basic vortex lines of the boundary layer shear are pinned to the front separation and stretched out down stream. Mason and Sykes conclude that in laminar flow over surface mounted objects, the generation of mean flows transferring momentum to the surface seem essential for a significant increase in the net momentum transfer.

Below we shall discuss the radiation of gravity waves in a stratified atmosphere but here we should note that the influence of stable stratification may be felt on shorter scales. In laminar flows on such short scales Mason and Sykes found that stable stratification suppressed the mean flow eddies giving a much reduced drag. For the turbulent atmospheric boundary layer on very short scales C_g is dramatically reduced by stable stratification. On larger scales the effect of stable stratification is not known.

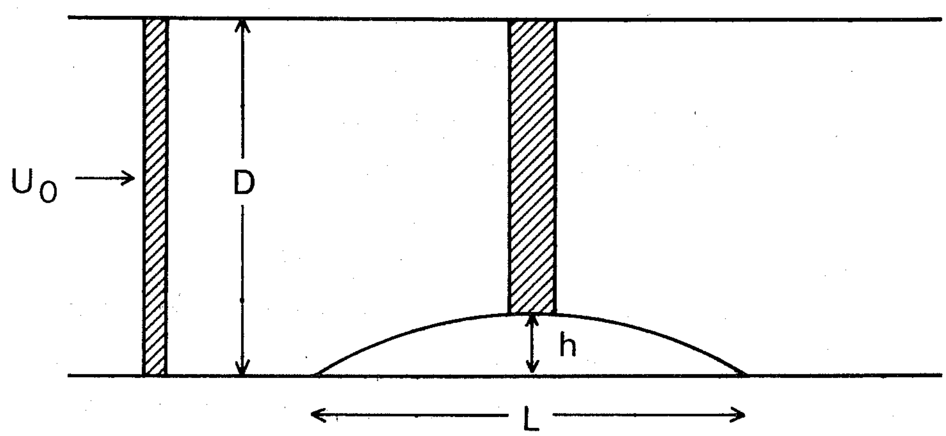
The difficult task of obtaining reliable forces for turbulent flow on these scales of order a kilometer is now vital.

b. Large scale "boundary layer pumping" drag

This should be mainly explicit even in large scale numerical models but, as we shall see, with a stable atmosphere, it can also be important on scales of order of hundreds of kilometers. The difficulty with this drag arises through the drag being proportional to the square of the height of the topography so that smoothed topography will not have the correct effect.

Below we give a "back of the envelope" derivation of the drag which illuminates its nature. A rigorous analysis giving the same result apart from a small numerical correction can be found in Mason and Sykes (1978).

Consider flow at small Rossby number $R = U/fL \ll 1$ (Mason and Sykes found it to occur provided $^{U}/fL \leqslant 1$) in bounded fluid of depth D where $D \ll L/R$ (Fig. 1).



For these parameters the relative vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ can be inferred from the conservation of

$$\frac{f + \zeta}{D} \quad \text{thus} \quad \frac{\partial v}{\partial x} \sim - \frac{fh}{D}$$

$$\text{and} \qquad v \sim - \frac{Lfh}{2\pi D}$$

The total energy in the disturbance above the topography is thus of order

$$\frac{1}{2}\rho v^2 \, L^2 \, D \sim \rho \frac{L^4 \, f^2 \, h^2}{8 \, \pi^2 \, D}$$

Assuming that dissipation is confined to the boundary layer we may integrate the momentum equations and using continuity find a pumping velocity

$$w = \frac{1}{\rho f} \left(\frac{\partial \tau y}{\partial x} - \frac{\partial \tau x}{\partial y} \right)$$

where τx and τy are the x and y components of surface stress. If the velocity perturbations due to the topography are $\ll U_o$ then for a turbulent boundary layer

$$\frac{\partial \tau y}{\partial x} \sim - \rho \, C_G \, U_o \frac{\partial v}{\partial x} \cos \alpha$$

where C_G is drag coefficient and α the angle of the surface stress to U_o.

Thus $w \sim - \dfrac{C_G \, U_o}{f} \quad \dfrac{\partial v}{\partial x}$

This pumping velocity has a sense which stretches the fluid filament on top of the topography back to a state of zero relative vorticity. The time scale on which this occurs is

$$\frac{h}{w} \sim \frac{h}{\frac{C_g U}{f} \cdot \frac{f \, h}{D}} \sim \frac{D}{C_G U} \quad (\text{assumed} > \frac{L}{u})$$

Now

work done = total energy/time scale on which energy is dissipated

$$F_s \cdot U_o = \rho \frac{L^4 \, f^2 \, L^2}{8 \, \pi^2 \, D} \Big/ \frac{D}{C_G U_o}$$

$$F_s = \rho \frac{L^4 \, f^2 \, h^2 \, C_G}{8 \, \pi^2 \, D^2}$$

As we shall see in the summary this force is significant for atmospheric scales of $\geqslant 10^7$ m.

In a stratified fluid at small Rossby number the vertical scale H associated with a horizontal disturbance L is

$$\sim \frac{L f}{2\pi N} \qquad \text{where } N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z}$$

Brunt Värsälä frequency

(see inertia-gravity wave dispersion relation below).

if $H < D$ the above argument follows through with H replacing D and we obtain

$$Fss = \frac{L^2 N^2 h^2 C_G}{2}$$

As we shall see below this force is significant for atmospheric scales of $10^6 - 10^7$ m.

3. Wave drags

a. Inertia-gravity waves

For scales between the short ones on which aerodynamic forces must be considered to the large scales on which boundarv pumping occurs, wave drag from inertia gravity waves is of potential importance. The parameterisation of such wave drags represents a most difficult challenge. The wave energy generated by the topography may be a complex function of wind and temperature structure, but worst of all the flow retardation arising from the wave generation is not confined to the boundary layer, but occurs where the waves are dissipated. This dissipation may frequently occur at "critical levels" in the atmosphere where the vertical propagation of the waves is halted.

Again we present an order of magnitude estimation of the drag effects. Consider the linearised inviscid inertia gravity wave equations (Queney 1947).

$$U_o \frac{\partial u}{\partial x} - fv = - \frac{\partial p}{\partial x}$$

$$U_o \frac{\partial v}{\partial x} + fu = - \frac{\partial p}{\partial y}$$

$$U_o \frac{\partial w}{\partial x} = \frac{\partial p}{\partial z} + \sigma; \quad \sigma = g \frac{\rho}{\bar{\rho}}$$

$$U_o \frac{\partial \sigma}{\partial x} - N^2 w = 0$$

Boundary condition $w = U_o \frac{\partial h}{\partial x}$ below and unbounded above. Solve by Fourier transform solution of the form

$$U = U^* (k,l) e^{imz}$$

where

$$m^2 = - \frac{N^2 k^2}{f^2} \frac{(1 - U_o^2 k^2/N^2)}{(1 - U_o^2 k^2/f^2)}$$

$U_o k/f = R$ the Rossby number and

$U_o k/N = F$ the Froude number thus.

$$m^2 = - \frac{N^2 k^2}{f^2} \frac{(1 - F^2)}{(1 - R^2)}$$

For $R < 1$ and $F < 1$

$$m = i Nk/f$$

evanescent vertical scale $\sim \frac{NL}{2\pi f}$ as used above - no wave drag occurs.

This is true for atmospheric scale greater than typically 600 Km.

For $R > 1$ and $F > 1$

$$m = ik$$

potential flow and no wave drag.

This is true for atmospheric scales of less than typically 6 Km.

For $R > 1$ and $F < 1$

$$m \approx \frac{N}{U}$$

gravity waves

For $R < 1$ and $F > 1$

$$m \approx k \frac{k U_o}{f} \approx k \cdot R$$

inertial waves - not appropriate for atmosphere since $N > f$ and vertical scale $\frac{1}{kR}$ very much greater than D.

Now

$$F_w \quad U = \frac{1}{2} \rho \; V^2 \qquad A \qquad C_g$$

work done	energy density	area radiating	group velocity

To obtain the group velocity take $\frac{\partial}{\partial t}$ instead of $U_o \frac{\partial}{\partial x}$ in the Queney equations and for $R \gg 1$ we obtain

$$\omega^2 = \frac{N^2 (k^2 + l^2)}{(m^2 + k^2 + l^2)}$$

and

$$\frac{\partial \omega}{\partial m} \sim \frac{U^2 \; 2\pi}{L \; N}$$

The vertical velocity produced by the topography

$$w \sim \frac{U_o \; h \; 2\pi}{L}$$

For the gravity waves with $F \ll 1$ from continuity horizontal velocities are greater

i.e. $U \sim w \quad \frac{m}{k} \sim N \; h$

Thus the energy is $\sim \frac{1}{2} \rho \, N^2 L^2$

and

$$F_G \quad U = \frac{1}{2} \rho \; N^2 \; h^2 \qquad L^2 \quad \frac{U^2 \; 2\pi}{LN}$$

$$F_G = \rho \, \pi \; U \; N \; L \; h^2$$

Again we have a drag force dependent on the square of the amplitude of the topography. This aspect of the results is dependent on our use of linear theory.

In the case of internal gravity waves the assumptions are

$\frac{h}{L} \ll 1$ and $\frac{U}{Nh} \gg 1$.

The restriction in slope is straightforward and $\frac{h}{L}$ is effectively limited to ~ 1 by flow separation. Numerical experiment by Mason and Sykes suggest that even in these cases linear theory gives a reasonable estimate of drag. The latter criteria is often more important. The Froude number $\frac{U}{Nh}$ is ratio of the fluid kinetic energy to the potential energy needed to lift fluid a height h. When

$\frac{U}{Nh} > 1$ fluid goes around rather than over the topography.

$\frac{U}{Nh}$ is generally ~ 1 for $h \sim 1$ kilometre though under stable boundary layer conditions it may be unity for smaller h.

In the case of the linearised theory of Ekman pumping drag given above the critical assumptions are

$\frac{h}{DR} \ll 1$ in the homogeneous case and

$\frac{U_o}{Nh} \gg 1$ again in the stratified case. Both of these criteria measure the magnitude of velocity perturbations relative to U_o and determine whether the flow goes over or around the topography. This was clearly indicated by laboratory experiments (Mason 1977) on flow in a rotating stratified fluid.

b. Other wave drags

The radiation of Rossby waves is outside the scope of this document but for perspective a crude estimation can be made. On these long scales motion is nearly two dimensional with waves being radiated horizontally.

For large topography we may calculate an upper bound by assuming the velocity perturbations are $\sim U_o$. The group velocity is $C_g \sim \frac{L^2\beta}{4\pi^2}$.

Thus

work done = energy density × area radiating × group velocity

$$F_R \quad U = \frac{1}{2} \rho U_o^2 \times L D \times \frac{L^2\beta}{4\pi^2}$$

$$F_R \quad = \frac{1}{8\pi^2} \rho U \quad L^3 D \beta .$$

This force is only important on atmospheric scales of $\geqslant 10^7$ m (N.B. the $8\pi^2$!).

We have not considered any possible radiation of other synoptic scale disturbances such as neutral or growing baroclinic waves. Unlike the other rough estimates given here a formal estimate is not available and it would be foolish to proceed very far. As an upper bound we may take a vertical scale $\frac{Lf}{2\pi N}$ and assume horizontal radiation with velocity U_o.

This gives

$$F_? = \frac{1}{2}\rho U_o^2 \; L^2 \; (\frac{f}{2\pi N})$$

which is generally significant but not dominant. In as far as this is probably a considerable over estimate such effects may not be very important in momentum transfer.

4. Summary

Here we compare the magnitude of estimates of force, as given above, with the stress due to a typical turbulent boundary layer, i.e. $C_g = 2.10^{-3}$ and stress $\tau_o = 2.10^{-3}\, U_g^2$. We have used $f = 10^{-4}\, s^{-1}$, $N = 10^{-2}\, s^{-1}$, $U = 10$ m s^{-1}, $D = 10^4$ and h and L as indicated in Table 1.

1. Aerodynamic

$$\frac{\tau_A}{\tau_o} = \frac{\frac{1}{2}\rho\, ChLU^2}{C_g\, U^2L^2} = \frac{1}{2} \cdot \frac{C}{C_g} \cdot \frac{h}{L}$$

2. Homogeneous-boundary layer pumping

$$\frac{\tau_S}{\tau_o} = \frac{\dfrac{L^2\, N^2\, h^2\, C_g}{8\pi^2\, D^2}}{C_g\, U^2\, L^2} = \frac{L^2\, f^2}{U^2} \cdot \frac{h^2}{D^2} \cdot \frac{1}{8\pi^2}$$

3. Stratified-boundary layer pumping

$$\frac{\tau_{ss}}{\tau_o} = \frac{\frac{L^2 N^2 h^2 C_g}{2}}{C_g U^2 L^2} = \frac{N^2 h^2}{2 U^2}$$

4. Gravity wave radiation

$$\frac{\tau_G}{\tau_o} = \frac{\pi U N L h^2}{C_g U^2 L^2} = \frac{\pi}{C_g} \cdot \frac{h}{L} \cdot \frac{Nh}{U}$$

5. Rossby wave radiation

$$\frac{\tau_R}{\tau_o} = \frac{\frac{U L^3 D}{8\pi^2}}{C_g U^2 L^2} = \frac{1}{8\pi^2} \cdot \frac{D}{L} \cdot \frac{L^2 \beta}{U}$$

TABLE 1

L/m	10^2	10^3	10^4	10^5	10^6	10^7
h/m	10	100	10^3	10^3	10^3	10^3
1. τ_A/τ_o	~1	~1	~1			
2. τ_S/τ_o						~1
3. τ_{ss}/τ_o					~1	
4. τ_G/τ_o			~1	~1		
5 τ_R/τ_o						~1

References

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