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THE ADJOINT EQUATION TECHNIQUE APPLIED TO METEOROLOGICAL PROBLEMS

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1. INTRODUCTION

The new approach to the problems of climate theory and long-range forecasting developed by academician G.I. Marchuk a few years ago has attracted the general attention of the meteorological community. This approach is based on the use of the properties of the adjoint to the equations of thermo and hydrodynamics of the atmosphere and ocean.

This report is essentially a detailed commentary on one of the chapters from Marchuk's book, "Numerical solution of the problems of atmosphere and ocean dynamics", 1974, which, unfortunately, is not yet available in English.

The adjoint-equation technique of the hydro and thermo dynamics allows us to construct functionals which represent the average, in space and time, of deviations of temperature, precipitation and some other fields from their climatic values. These functionals depend on the initial values and boundary conditions of mathematical models describing the evolution of physical processes in the atmosphere, ocean and active layer of the soil. Here, the variables in the expressions for the functionals have certain weights, which are determined by the solution of the adjoint problem. These solutions of the adjoint problems are the influence functions for the particular regions and certain time intervals.

A preliminary analysis shows that, with the increase of the time interval, the influence of the initial data decreases considerably. Then, after sufficient time, the influence of the processes in the atmosphere and soil also diminishes because they have comparatively short characteristic relaxation times. Eventually the ocean comes to the fore as a main decisive determining factor in forming the large scale motions and long lasting changes in the patterns of atmospheric processes over a given region of the earth.

These conclusions were illustrated by the first and comparatively simple numerical experiment based on climate and model data (G.I. Marchuk, Yu. N.Skiba, 1976). It appears that the influence function representing the adjoint function for the temperature calculated for the European part of the USSR for November has a maximum value of 6 to 8 months before this month in the region where the Gulf Stream originates. For Northern America, the region of origin of the Kuro-Shio is essential along with the tropical part of the Pacific and the northern part of the Indian Ocean.

These results formed the basis of the Soviet Union proposal for the international long term research project aimed to study the processes of interaction between the atmosphere and the ocean for the purposes of the development of climate theory and long-range forecasting.

The adjoint equation technique gives the possibility of obtaining both qualitative and quantitative assessments of the importance of different regions of the earth for such a research project. The expensive and limited resources for such a programme might be used in the most effective way by concentrating them in the regions of the most active interaction between the atmosphere and the ocean.

Such regions it seems are the regions of the birth of the powerful ocean currents, upwelling regions and the regions of intensive oceanic convection.

In the last few years, scientists in the Siberian Computer Centre and in the Hydrometcentre of USSR have carried out a number of projects directed to the study of the possible applications of the adjoint-equations technique for a broad class of meteorological problems from weather forecasting to the problems of environment protection and rational usage of natural resources.

Since the review of these works goes beyond the scope of the present report, the author gives just a list of publications, which the author cannot, of course, pretend to be a complete one.

2. THE DERIVATION OF ADJOINT EQUATIONS AND SOME OF THE FUNCTIONALS

In this section we shall show, using simple examples, how one derives adjoint equations, how one should solve them, what meaning they have and how, using them, one may construct different functionals for the purposes of diagnosis and weather forecasting.

First we need to introduce a few definitions.

Let us define the inner produce in the Hilbert space as follows.

$$(g,h)_D = \sum_{i=1}^{n} \int_D g_i \, h_i \, dD$$

Here g_i and h_i are components of vector functions g and h respectively, n is the dimension of these vectors and D the spatial domain over which the equations of thermo and hydrodynamics are integrated.

For the adjoint technique we shall need a more general definition of the inner product:

$$(g,h)_{DxT} = \sum_{i=1}^{n} \int_{t_o}^{t_1} \int_D g_i \, h_i \, dD \, dt$$

Where T is a temporal domain, t_o, t_1, the interval of integration in time.

Let us notice that in the spherical σ-system of coordinates, the inner product has the following form.

$$(g,h)_D = \sum_{i=1}^{n} \int_\sigma \int_\lambda \int_\phi \pi a^2 \cos\phi \; g_i \; h_i \; d\sigma \; d\lambda \; d\phi$$

where π is the pressure at the earth's surface.

Let A represent an operator transforming the vector function h in the Hilbert space into some new vector function belonging to the same space, and define the adjoint of A, A^*, by

$$(g, Ah)_D = (A^* g, h)_D$$

<u>Examples</u>

(a) Let us begin with the simplest case of a barotropic atmosphere:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv + RT \frac{\partial \Phi}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu + RT \frac{\partial \Phi}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Here u and v are components of the velocity vector, Φ - the deviation of relative "pressure" from the standard one, and T - the average temperature in the domain D. Let D be a rectangle and on its boundaries periodicity conditions are prescribed.

Let us now introduce the vector of the solution, ϕ, the adjoint vector ϕ^* and matrices A and B:

$$\phi = \begin{bmatrix} u \\ v \\ RT\Phi \end{bmatrix}, \quad \phi^* = \begin{bmatrix} u^* \\ v^* \\ RT\Phi^* \end{bmatrix}, \quad A = \begin{bmatrix} \Lambda & -f & \frac{\partial}{\partial x} \\ f & \Lambda & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\Lambda \equiv u \frac{\partial \phi}{\partial x} + v \frac{\partial}{\partial y}$

Now we can write the system (1) in the operator form:

$$B \frac{\partial \phi}{\partial t} + A\phi = 0, \quad \phi_{t=t_o} = \phi_o \qquad (2)$$

Let us assume that the problem adjoint to (2) is

$$B^* \frac{\partial \phi^*}{\partial t} + A^* \phi^* = 0 \qquad (3)$$

Now, making use of the Lagrangian identity

$$(A\,\phi,\ \phi^*)_{DxT} = (A^*\phi^*, \phi)_{DxT} \qquad (4)$$

we shall derive expressions for B^* and A^*.

To this end, let us write the inner product $(A\phi, \phi^*)_{DxT}$ in the form:

$$(A\phi,\phi^*)_{DxT} = \int_{t_o}^{t_1} \int_D \Big((\Lambda u - fv + RT \frac{\partial \Phi}{\partial x}) u^* + (fu + \Lambda v + RT \frac{\partial \Phi}{\partial y}) v^* + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) RT\, \phi^* \Big)\, dD\, dt \qquad (5)$$

also we rewrite separately the expression

$$\int_{t_o}^{t_1} \int_D (\Lambda u \,.\, u^* + \Lambda v \,.\, v^*) dD\, dt = \int_{t_o}^{t_1} \int_D (u \frac{\partial u}{\partial x} u^* + v \frac{\partial u}{\partial y} u^* + u \frac{\partial v}{\partial x} v^* + v \frac{\partial v}{\partial y} v^*) dD\, dt \qquad (6)$$

We then integrate by parts the first two terms on the right hand side of Equation (6)

$$\int_{t_o}^{t_1} \int_D (u \frac{\partial u}{\partial x} u^* + v \frac{\partial u}{\partial y} u^*)\, dD\, dt =$$

$$= \int_{t_o}^{t_1} \Big(\int_y dy \int_x u \frac{\partial u}{\partial x} u^* \, dx + \int_x dx \int_y v \frac{\partial u}{\partial y} u^* \, dy \Big)\, dt =$$

$$= \int_{t_o}^{t_1} \Big(\int_y dy\, (uu^* u \Big|_o^{2\pi} - \int_x u \frac{\partial uu^*}{\partial x} dx) + \int_x dx\, (vu^* u \Big|_o^{2\pi} - \int_y u \frac{\partial vu^*}{\partial y} dy) \Big)\, dt$$

In this expression

$$uu^* u \Big|_o^{2\pi} = vu^* u \Big|_o^{2\pi} = 0$$

due to periodicity conditions on the boundary of D.

Now,
$$-\int_{t_o}^{t_1}\int_y dy \int_x u\frac{\partial u\, u^*}{\partial x} dx\, dt = -\int_{t_o}^{t_1}\int_y dy \int_x (uu\frac{\partial u^*}{\partial x} + uu^*\frac{\partial u}{\partial x})\, dx\, dt$$

$$-\int_{t_o}^{t_1}\int_x dx \int_y u\frac{\partial vu^*}{\partial y} dy\, dt = -\int_{t_o}^{t_1}\int_x dx \int_y (uv\frac{\partial u^*}{\partial x} + uu^*\frac{\partial v}{\partial y})\, dy\, dt\ ;$$

$$-\int_{t_o}^{t_1}\Big(\int_y dy \int_x uu^*\frac{\partial u}{\partial x} dx + \int_x dx \int_y uu^*\frac{\partial v}{\partial y} dy\Big)\, dt =$$

$$-\int_{t_o}^{t_1}\int_D uu^*\ (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\ dD\, dt = 0 \quad ,$$

since u and v satisfy the continuity equation in (1).

So,
$$\int_{t_o}^{t_1}\int_D (u\frac{\partial u}{\partial x}u^* + v\frac{\partial u}{\partial y}u^*)\ dD\, dt = -\int_{t_o}^{t_1}\int (u\frac{\partial u^*}{\partial x}u + v\frac{\partial u^*}{\partial y}u)\ dD\, dt\ .$$

In the same way, we may show that

$$\int_{t_o}^{t_1}\int_D (u\frac{\partial v}{\partial x}v^* + v\frac{\partial v}{\partial y}v^*)\ dD\, dt = -\int_{t_o}^{t_1}\int_D (u\frac{\partial v^*}{\partial x}v + v\frac{\partial v^*}{\partial y}v)\ dD\, dt.$$

Hence,

$$\int_{t_o}^{t_1}\int_D(\Lambda u.u^* + \Lambda v\cdot v^*)\ dD\, dt = -\int_{t_o}^{t_1}\int_D(\Lambda u^*\cdot u + \Lambda v^*\cdot v)\ dD\, dt \qquad (7)$$

We can also write

$$\int_{t_o}^{t_1}\int_D (-fv\cdot u^* + fu\, v^*)\ dD\, dt = \int_{t_o}^{t_1}\int_D (fv^*\cdot u - fu^*\cdot v)\ dD\, dt \qquad (8)$$

Integrating by parts the remainder of (5) and using periodicity conditions, we get:

$$\int_{t_o}^{t_1}\int_D RT\frac{\partial\Phi}{\partial x}\ \Phi^*\ dD\, dt = -\int_{t_o}^{t_1} RT\int_D \frac{\partial u^*}{\partial x}\ \Phi\, dD\, dt$$

$$\int_{t_o}^{t_1}\int_D RT \frac{\partial \Phi}{\partial y} v^* \, dD\, dt = -\int_{t_o}^{t_1} RT \int_D \frac{\partial v^*}{\partial y} \cdot \Phi \, dD\, dt$$

$$\int_{t_o}^{t_1}\int_D \frac{\partial u}{\partial x} RT\, \Phi^* \, dD\, dt = -\int_{t_o}^{t_1} RT \int_D \frac{\partial \Phi^*}{\partial x} \cdot u \, dD\, dt ,$$

$$\int_{t_o}^{t_1}\int_D \frac{\partial v}{\partial y} RT\, \Phi^* \, dD\, dt = -\int_{t_o}^{t_1} RT \int_D \frac{\partial \Phi^*}{\partial y} \cdot v \, dD\, dt$$

Now, we have

$$(A\phi,\ \phi^*)_{DxT} = \int_{t_o}^{t_1}\int_D \left(-\Lambda u^* + f v^* - RT\frac{\partial \Phi^*}{\partial x}\right) u \, +$$

$$+ \left(-f u^* - \Lambda v^* - RT\frac{\partial \Phi^*}{\partial y}\right) v + \left(-\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y}\right) RT\,\Phi \Big)\, dD\, dt =$$

$$= (A^* \phi^*, \phi)_{DxT} , \quad \text{where} \quad A^* = -A \tag{9}$$

Analogously

$$\left(B\frac{\partial \phi}{\partial t},\ \Phi^*\right)_{DxT} = \int_D \int_{t_o}^{t_1} \left(\frac{\partial u}{\partial t} u^* + \frac{\partial v}{\partial t} v^* + R^2T^2 \frac{\partial \Phi}{\partial t}\Phi^*\right) dt\, dD$$

$$= -\int_D \int_{t_o}^{t_1} \left(\frac{\partial u^*}{\partial t} u + \frac{\partial v^*}{\partial t} v + R^2T^2 \frac{\partial \Phi^*}{\partial t}\Phi\right) dt\, dD = \left(B^* \frac{\partial \Phi^*}{\partial t},\ \Phi\right)_{DxT} \tag{10}$$

Where $B^* = -B$ apart from the differences in the initial values for the direct and adjoint problems.

If we have zero initial conditions for both problems $B^* = -B$.

We have thus shown that, for the direct problem (2), there is the corresponding adjoint problem:

$$B^* \frac{\partial \phi^*}{\partial t} + A^* \phi^* = 0 \quad , \ \phi^*_{t=t_1} = \phi^*_1 , \tag{11}$$

where $B^* = -B$, $A^* = -A$.

The sign " - " here stands for the fact that in the adjoint problem, the movement and development of the processes is going in the opposite direction in space and time than in the direct problem (2).

This means that while solving the problem (11) one must formulate initial conditions on the right boundary of the time interval (t_o, t_1).

We shall show later that the backward movement and evolution of the processes in the adjoint problem has a profound physical and informational sense.

Let us write the direct and adjoint problems in operator form:

$$B \frac{\partial \phi}{\partial t} + A\phi = 0 \ , \quad \phi_{t=t_o} = \phi_o \tag{12}$$

$$-B \frac{\partial \phi^*}{\partial t} + A^* \phi^* = 0 \ , \quad \phi^*_{t=t_1} = \phi^*_1 \tag{13}$$

We shall now take the inner product of (12) with ϕ^* and (13) with ϕ and subtract the second from the first:

$$(\phi^*, B\frac{\partial \phi}{\partial t})_{DxT} + (\phi, B \frac{\partial \phi^*}{\partial t})_{DxT} = 0$$

$$\frac{\partial}{\partial t} (\phi^*, B\phi)_{DxT} = 0$$

$$\int_D (\phi^*_o , B\phi_o) \, dD = \int_D (\phi^*_1 , B \phi_1) \, dD \ ,$$

or, in component form:

$$\int_D (u_1 u^*_1 + v_1 v^*_1) dD = \int_D (u_o u^*_o + v_o v^*_o) \, dD \tag{14}$$

Let us choose the initial conditions for the adjoint problem (13) as follows:

$$\phi^*_{t=t_1} = \phi_{t=t_1} = \phi_1 \tag{15}$$

Then we arrive at the law of "conservation of kinetic energy"

$$\int_D (u_1^2 + v_1^2) \, dD = \int_D (u_o^2 + v_o^2) \, dD \tag{16}$$

or $\quad \int_D E_1 \, dD = \int_D E_o \, dD$

The equality (16) expresses the fact also that there is the complete convertibility of the solution for the problems (12) and (13). This means that if one solves the adjoint problem (13) with initial conditions (15), one arrives at the solution which is exactly the same as initial data for the direct problem (12). Of course, while solving (13), one should use on each time step values of u and v for the operator Λ as a solution of the direct problem and T must be the same constant in both problems.

(b) Let us consider now the problem (2) with the addition of a turbulent viscosity:

$$\frac{\partial u}{\partial t} + \Lambda u - fv + RT\frac{\partial \Phi}{\partial x} - \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) = 0$$

$$\frac{\partial v}{\partial t} + \Lambda v + fu + RT\frac{\partial \Phi}{\partial y} - \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

In this case

$$A = \begin{bmatrix} \Lambda - \frac{\partial}{\partial x}\mu\frac{\partial}{\partial x} & -f & \frac{\partial}{\partial x} \\ f & \Lambda - \frac{\partial}{\partial y}\mu\frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{bmatrix}$$

Let us obtain the system adjoint to the system of equations (17) using the technique described above. Since the only difference from problem (2) now is with the turbulence terms, we shall follow the transformations of one of them in the inner product (5) :

$$-\int_x \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) u^* \, dx = -\mu\frac{\partial u}{\partial x} u^* \Big|_o^{2\pi} + \int_x \frac{\partial u}{\partial x}\, \mu\frac{\partial u^*}{\partial x}\, dx =$$

$$= \mu\frac{\partial u^*}{\partial x}\, u \Big|_o^{2\pi} - \int_x \frac{\partial}{\partial x}\, \mu\frac{\partial u^*}{\partial x}\, u\, dx = -\int_x \frac{\partial}{\partial x}\left(\mu\frac{\partial u^*}{\partial x}\right) u\, dx$$

Analogously,

$$-\int_y \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) v^* dy = -\int_y \frac{\partial}{\partial y}\left(\mu\frac{\partial v^*}{\partial y}\right) v dy$$

We have thus seen that in the case of the system with viscosity described by the turbulent terms having second derivatives these terms do no change sign while we derive the formulae for the adjoint problem. This is due to the fact that we have employed integration by parts twice.

This means, in this case, $A^* \neq -A$.

Indeed, turbulent terms in the equations describe diffusion and therefore they must have the same sign independent of the direction of the processes in time and space.

The operator A^* for the problem adjoint to (2) has the form:

$$A^* = \begin{bmatrix} -\Lambda - \frac{\partial}{\partial x}\mu\frac{\partial}{\partial x} & f & -\frac{\partial}{\partial x} \\ -f & -\Lambda - \frac{\partial}{\partial y}\mu\frac{\partial}{\partial y} & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \end{bmatrix}$$

Up to this point, while deriving the formulae for the adjoint equations, we have enjoyed the convenience of periodicity conditions on the boundaries. Let us now make a next step and analyse a more complicated baroclinic problem which has derivatives along the vertical coordinate.

(c) We shall write the system of equations for the atmosphere in adiabatic and quasi-stationary approximations in the p-coordinate system:

$$\frac{\partial u}{\partial t} + \Lambda u - fv + \frac{\partial \Phi}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + \Lambda v + fu + \frac{\partial \Phi}{\partial y} = 0$$

$$\frac{\partial \Phi}{\partial p} + \frac{R}{p} T = 0 \qquad (18)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{\partial T}{\partial t} + \Lambda T - \frac{R}{c_p p} \bar{T} \omega = 0$$

Here $\Lambda = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$, and $\bar{T}$ = const.

Let us introduce again the vectors ϕ and ϕ^* and the matrices A and B.

$$\phi = \begin{bmatrix} u \\ v \\ \omega \\ \phi \\ T \end{bmatrix}, \quad \phi^* = \begin{bmatrix} u^* \\ v^* \\ \omega^* \\ \phi^* \\ T^* \end{bmatrix}, \quad A = \begin{bmatrix} \Lambda & -f & 0 & \frac{\partial}{\partial x} & 0 \\ f & \Lambda & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & \frac{-R}{p} & 0 & \frac{c_p}{\bar{T}}\Lambda \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{c_p}{\bar{T}} \end{bmatrix}$$

We add boundary conditions along p to the boundary conditions along horizontal coordinates:

$$\left.\begin{aligned} \omega = 0 \quad & \text{at} \quad p = 0 \\ \omega = 0 \quad & \text{at} \quad p = P \end{aligned}\right\} \tag{19}$$

Scrutinising the structure of the operator A and using the experience acquired in deriving the adjoint equations, we can show that the adjoint operation has the form

$$A^* = \begin{bmatrix} -\Lambda & f & 0 & -\frac{\partial}{\partial x} & 0 \\ -f & -\Lambda & 0 & -\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial p} & -\frac{R}{p} \\ -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & \frac{R}{p} & 0 & -\frac{c_p}{\bar{T}}\Lambda \end{bmatrix}$$

i.e. $A^* = -A$.

Let us write the problem (18) and its adjoint in the operator form:

$$B\,\frac{\partial \phi}{\partial t} + A\phi = 0 \tag{20}$$

$$-B\,\frac{\partial \phi^*}{\partial t} + A^*\,\phi^* = 0 \tag{21}$$

with

$$\left.\begin{aligned} \omega^* = 0 \quad & \text{at} \quad p = 0 \\ \omega^* = 0 \quad & \text{at} \quad p = P \end{aligned}\right\} \tag{22}$$

We shall take the inner product of (20) with ϕ^*, (21) with Φ and subtract the second from the first:

$(B\phi_1, \phi_1^*)_D = (B\phi_o, \phi_o^*)_D$, or in component form:

$$\int_D (u_1\, u_1^* + v_1\, v_1^* + \frac{c_p}{\bar{T}}\; T_1\, T_1^*)\, dD = \int_D (u_o u_o^* + v_o v_o^* + \frac{c_p}{\bar{T}}\, T_o T_o^*)\, dD \tag{23}$$

Let us set $u_1^* = u_1$, $v_1^* = v_1$, $T_1^* = T_1$

and define the total energy: $\pi = \frac{1}{2}\,(u^2 + v^2 + \frac{c_p}{\bar{T}}\,T^2)$;

then $\int_D \pi_1 dD = \int_D \pi_o dD$,

and we have arrived at the law of total energy conservation.

Let us now set $u_1^* = 0$, $v_1^* = 0$ and $T_1^* = \frac{\bar{T}}{c_p}\,\delta\,(x - x_o,\ y - y_o,\ p = p_o)$ (24)

This means that T_1^* is different from zero just at the point with the coordinates $(x_o,\ y_o,\ p_o)$. Then from (23) we have the expression for the temperature at the given point of space $(x_o,\ y_o,\ p_o)$ at the moment t_1:

$$T_1\ (x_o, y_o, p_o) = \int_D (u_o\, u_o^* + v_o\, v_o^* + \frac{c_p}{\bar{T}}\,T_o\, T_o^*)\, dD \qquad (25)$$

Having set the initial values (24) for the adjoint problem (21), by this act from the infinity of possible solutions ϕ^* we have picked up the only one ϕ_o^* which describes the process of propagation and evolution of the signal from the point $(x_o,\ y_o,\ p_o)$ with the velocities equal to in value but opposite in direction to those we have obtained as a result of solution of the direct problem (20). The evolution itself in this case is backward in time.

As a result of the solution of the adjoint problem (21) with the boundary conditions (22) and initial conditions (24), we have found the time and space distribution of the field of the vector function ϕ^* (x, y, p, t).

When we use in (25) values of the components of the vector-function ϕ^* at the moment of time t_o, we have the expression for the temperature at the given point in space through the initial fields of u_o, v_o, T_o weighted with the corresponding weights u_o^*, v_o^*, T_o^*.

It may happen that, in certain regions of the domain D, u_o^*, v_o^*, T_o^* reach maximum values and then these regions will give maximum input into the forming of temperature at the point $(x_o,\ y_o,\ p_o)$ at the moment of time t_1. On the other hand in some other regions u_o^*, v_o^*, T_o^* may have minimal values and in this case the information about initial fields u_o, v_o, T_o from such regions has no real value.

Hence we arrive at the main conclusion: the solution of the adjoint problem ϕ^* has a sense of the value of the information.

(d) We shall now make the last step in one consideration of the simplest examples, and proceed with the study of the diabatic problem:

$$\frac{\partial u}{\partial t} + \Lambda u - fv + \frac{\partial \Phi}{\partial x} - \mu \Delta u = 0$$

$$\frac{\partial v}{\partial t} + \Lambda v + fu + \frac{\partial \Phi}{\partial y} - \mu \Delta v = 0$$

$$\frac{\partial \Phi}{\partial p} + \frac{R}{p} T = 0 \tag{26}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{c_p}{\bar{T}} \frac{\partial T}{\partial t} + \frac{c_p}{\bar{T}} \Lambda T - \frac{R}{p} \omega - \frac{c_p}{\bar{T}} \left(\frac{\partial}{\partial p} \nu \frac{\partial T}{\partial p} + \mu_T \Delta T\right) = 0$$

With boundary conditions in p:

$$\omega = 0 \quad \frac{\partial T}{\partial p} = 0 \qquad \text{at } p = 0$$
$$\omega = 0 \quad \frac{\partial T}{\partial p} = \alpha_s \cdot (T - T_s) \quad \text{at } p = P \tag{27}$$

Where T_s is a temperature of the underlaying surface, and α_s is a heat transfer coefficient.

The operator A for this problem is as follows:

$$A = \begin{bmatrix} \Lambda - \mu\Delta & -f & 0 & \frac{\partial}{\partial x} & 0 \\ f & \Lambda - \mu\Delta & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\frac{R}{p} & 0 & \frac{c_p}{\bar{T}} (\Lambda - \frac{\partial}{\partial p} \nu \frac{\partial}{\partial p} - \mu_T \Delta) \end{bmatrix}$$

The problem (26) differs from the previous adiabatic problem (18) only through the additional term in the thermodynamic equation, describing the vertical turbulent heat flux in the atmosphere.

Let us follow the transformations of this particular term in the inner product:

$$\int_p \frac{\partial}{\partial p} \nu \frac{\partial T}{\partial p} \cdot T^* \, dp = T^* \nu \frac{\partial T}{\partial p} \Big|_o^P - \int_p \frac{\partial T}{\partial p} \cdot \nu \cdot \frac{\partial T^*}{\partial p} \, dp =$$

$$= - T_p^* \; \nu_p \, \alpha_s \, (T_p - T_s) - T\nu \frac{\partial T^*}{\partial p} \Big|_o^P + \int_p \frac{\partial}{\partial p} \nu \frac{\partial T^*}{\partial p} \, T \, dp$$

Now it is time to make a choice for the boundary conditions for the adjoint problem:

$$\frac{\partial T^*}{\partial p} = 0 \qquad \text{at } p = 0$$

$$\frac{\partial T^*}{\partial p} = \alpha_s T^* \qquad \text{at } p = P \tag{28}$$

Then

$$\int_p \frac{\partial}{\partial p} \nu \frac{\partial T}{\partial p} T^* \, dp = \int_p \frac{\partial}{\partial p} \nu \frac{\partial T^*}{\partial p} \, T \, dp + R^1 ,$$

where the residual R^1 is:

$$R^1 = -T_p^* \, \nu_p \, \alpha_s \, (T_p - T_s) + T_p \, \nu_p \, \alpha_s \, T_s^* = \nu_p \, \alpha_s \, T_s \, T_p^* \tag{29}$$

$$A^* = \begin{bmatrix} -\Lambda-\mu\Delta & f & 0 & -\frac{\partial}{\partial x} & 0 \\ -f & -\Lambda-\mu\Delta & 0 & -\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial p} & -\frac{R}{p} \\ -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & \frac{R}{p} & 0 & \frac{c_p}{\bar{T}} \left(-\Lambda - \frac{\partial}{\partial p} \nu \frac{\partial}{\partial p} - \mu\Delta\right) \end{bmatrix}$$

Now, when we have non-homogeneous boundary conditions, the Lagrangian identity is satisfied with the residual R^1 :

$$(A \; \phi, \phi^*)_{DxT} = (A^* \; \phi^*, \phi)_{DxT} + \frac{c_p}{\bar{T}} \int_{t_o}^{t_1} R^1 \, dt$$

This is a mathematical statement of the fact that the system of equations (26) describes an unclosed physical system: there is exchange of energy between the atmosphere and underlaying surface in this system. Naturally, the law of energy conservation in this case is not valid any more.

Let us write the problem (26) and corresponding adjoint problem in operator form and continue our analysis:

$$B \frac{\partial \phi}{\partial t} + A\phi = 0 \;, \qquad \phi_{t=t_o} = \phi_o \tag{30}$$

$$-B \frac{\partial \phi^*}{\partial t} + A^* \phi^* = 0 \;, \; \phi^*_{t=t_1} = \phi^*_1 \tag{31}$$

The operators A and A^* include the description of the boundary conditions.

Take the inner product of (30) with ϕ^* and (31) with ϕ and subtract the second from the first:

$$\int_D (u_1 u_1^* + v_1 v_1^* + \frac{c_p}{\bar{T}} T_1 T_1^*) \, dD =$$

$$= \int_D (u_o u_o^* + v_o v_o^* + \frac{c_p}{\bar{T}} T_o T_o^*) \, dD + R$$

where

$$R = \int_{t_o}^{t_1} \int_S \nu_p \frac{c_p}{\bar{T}} \alpha_s T_s T_p^* \, ds \, dt$$

Now, the choice of initial conditions for the adjoint problem (31) is as follows:

$$u_1^* = 0, \; v_1^* = 0, \; T_1^* = \frac{\bar{T}}{c_p \Sigma} \text{ if } x,y \in G, \text{otherwise } T_1^* = 0 \tag{33}$$

where Σ is the area of the region G on the surface S which is interesting for us in some respect.

Let us be interested in the average value of the temperature in the region G at the moment of time t_1; we then obtain from (32):

$$\bar{T}_1^G = \int_D (u_o u_o^* + v_o v_o^* + \frac{c_p}{\bar{T}} T_D T_D^*) \, dD + \int_{t_o}^{t_1} \int_s \nu_p \frac{c_p}{\bar{T}} \alpha_s T_s T_p^* \, ds \, dt \tag{34}$$

The expression (34) has a clear physical sense: the average temperature $\bar{T}_1^G$ over the surface in the region G is formed with the initial data u_o, v_o, T_o with the weights u_o^*, v_o^*, T_o^* and with interaction between the atmosphere and the underlaying surface on the time interval (t_o, t_1) with the weight T_p^* (x, y, t).

Since the equation (31) has turbulence viscosity terms, one may expect that, if the time interval is big enough, u_o^*, v_o^* and T_o^* will dissipate eventually and (34) will be left with only the term describing interaction of atmosphere with the underlaying surface.

As a rough approximation, one can believe α_s over the ocean is much bigger than over the continents. Then in the long run, (34) describes mostly the input from the oceans into the forming of the average temperature $\bar{T}_1^G$ over the given region G at the moment of time $t = t_1$.

One can arrive at the expression (34) in a slightly different way:

Let us put $u_1^* = v_1^* = T_1^* = 0$,

but change boundary condition (28):

$$\frac{\partial T^*}{\partial p} = \alpha_s T_p^* + f^* \quad \text{at } p = P \tag{35}$$

where

$$f^* = \frac{\bar{T}}{c_p \nu_p \Sigma} \delta (t - t_1) \text{ for } x, y \in G, \text{ and } f^* = 0 \text{ otherwise}$$

Then the residual R^1 in (29) has the form:

$$R^1 = \nu_p \alpha_s T_s T_p^* + \frac{\bar{T}}{c_p \nu_p \Sigma} T_p \quad \text{and}$$

we obtain the same expression (34) for $\bar{T}_1^G$.

This means it is all the same if one sets up T_1^* at the initial moment t_1 on the surface G or puts zero initial values but has an instantaneous source in the boundary condition (35).

If we are interested not only in the average temperature over given region G but also in the average value over the time interval $(t_1 - t, t_1)$, we should put the source f^* in the form:

$$f^* = \frac{\bar{T}}{c_p \nu_p} \xi^* (x,y) \cdot \eta^* (t) \quad , x,y \in G, t \in [t_1 - \Delta t, t_1] \, ,$$

where the only condition we impose on the functions $\xi^*(x,y)$ and $\eta^*(t)$ is the normalizing condition:

$$\int_G \xi^*(x,y)\, ds = 1, \quad \int_{t_1-\Delta t}^{t_1} \eta^*(t)\, dt = 1.$$

Then in (34) $\overline{T}_1^G$ will be replaced by

$$\overline{T}_G = \int_{t_1-\Delta t}^{t_1} \int_G T\, ds \quad dt$$

(e) Now we have enough experience with the adjoint equation technique to proceed with some generalisations.

Consider the system of equations describing the climate of the atmosphere and the corresponding adjoint system of equations.

$$B\frac{\partial\phi}{\partial t} + A\phi = f$$
$$-B\frac{\partial\phi^*}{\partial t} + A^*\phi^* = 0 \qquad (36)$$

We have shown already that

$$(B\phi_1, \phi_1^*)_D - (B\,\phi_o, \phi_o^*)_D = (\phi^*, f)_{DxT} + R \quad , \qquad (37)$$

where the residual R appears because of the nonhomogeneous boundary conditions and describes the sources and sinks of energy in the climatic system.

Next, write the system of equations for the real atmosphere and join to this system the adjoint problem for the climate from (36)

$$B\frac{\partial\phi'}{\partial t} + A'\phi' = f'$$
$$-B\frac{\partial\phi^*}{\partial t} + A^*\phi^* = 0$$

Here

$$\phi' = \phi + \delta\phi, \quad A' = A + \delta A, \quad f' = f + \delta f.$$

$\delta\phi$, δA, δf are the deviations from the climate and are not necessarily small. Indeed, they may well be of the same order as the climate values of ϕ and f themselves.

Taking the inner product of the first equation with ϕ^* and the second with ϕ_1 we obtain:

$$(B\phi_1,\phi_2^*)_D - (B\phi_o,\phi_o^*)_D + (A'\phi',\phi^*)_{DxT} - (A^*\phi^*,\phi')_{DxT} = (f',\phi^*)_{DxT}$$

$$(B(\phi_1 + \delta\phi_1),\phi_1^*)_D - (B(\phi_o + \delta\phi_o),\ \phi_o^*)_D + \big((A + \delta A)(\phi + \delta\phi),\ \phi^*\big)_{DxT} =$$

$$= (A^*\phi^*,(\phi + \delta\phi))_{DxT} + \big((f + \delta f),\ \phi^*\big)_{DxT}\ .$$

Making use of (37) we have:

$$(B\ \ \delta\phi_1,\ \phi_1^*)_D - (B\ \delta\phi_o,\ \phi_o^*)_D + (A\ \delta\phi,\ \phi^*)_{DxT} + (\delta A\ \phi',\ \phi^*)_{DxT} =$$

$$= (A^*\ \phi^*,\ \delta\phi)_{DxT} + (\delta f,\ \phi^*)_{DxT}$$

It is evident that $(A\ \delta\phi,\ \phi^*)_{DxT} - (A^*\phi^*,\ \delta\phi)_{DxT} = \delta R$,

where δR is a residual describing the deviations of the sources and sinks of energy from their climatic values.

Hence we finally arrive at:

$$(B\ \delta\phi_1,\ \phi_1^*)_D - (B\ \delta\phi_o,\ \phi_o^*)_D + \int_{t_o}^{t_1} (\delta A\phi',\ \phi^*)_D\ dt =$$

$$= \int_{t_o}^{t_1} (\delta f,\ \phi^*)_D\ dt + \delta R \qquad (38)$$

In this expression, ϕ^* is the result of the solution of the adjoint climatic problem. In general, the operator A can encompass a description not only of atmospheric processes but also of those in the ocean, continents and upper and boundary layers of the atmosphere. The more complete climate information used while calculating ϕ^* the more successful will be the applications of (38).

The influence functions ϕ^* are calculated with use of climate information only once but for every given region and time of year separately. The sizes of such regions and the length of time intervals are chosen to be in correspondence with the goals for which these influence functions are calculated.

The longer the lead time of the forecast, the bigger the time interval. For example, if one has in mind the forecast for a season, one should use the information covering the whole of a year, or 6-8 months at least, and the time resolution must be not less than one month. Unfortunately, in most cases, we have to content ourselves with climate information on a monthly basis.

The choice of regions for which to calculate the influence functions is an interesting and important separate problem. Such regions should be of certain optimal sizes and react to external forcings as a climatic whole. If one has seasonal forecasts in mind, one could separate such regions as Western Europe, Eastern Europe, the European part of USSR, South-West Siberia and Northern Kazackstan, USA, India or Australia. But of course some objective criteria must be developed in order to define the boundaries of such regions.

In this case, the calculation of the influence functions, which requires heavy computer resources, is carried out only once for each region and every season of a year using the climatic data. The results might be stored and archived on magnetic tapes. Essentially it is one of the ways to develop and archive climate information.

Having such information, one may construct many different functionals, such as the anomalies of temperature, precipitation and pressure over certain regions. It is essential that the calculation of such functionals, using formulae of the kind (38), should require much smaller computer facilities than solution of the direct and adjoint problems (36) and can be carried out on very modest computers in regional weather centres.

Of course we face many difficult problems to solve; for instance, the separation of the operator δA is not at all trivial.

The bigger the time interval (t_o, t_1), the more important is the role played by the ocean. Therefore, it is necessary to have reliable and representative oceanographic information on the World Ocean, at any rate on a monthly basis. We need a mathematical model of the World Ocean on the same level as the best contemporary atmospheric models. More than this, it is necessary to have coupled models of the Atmosphere - Ocean - Continent interaction.

General perturbation theory has brought us to the result (38) and this result might be effectively used for diagnostic researches, for the assessments of the skill of the numerical models used for medium and long-range forecasting.

The above approach may be a useful basis when one plans the development of the World Weather Watch System and especially while planning global atmospheric and oceanographic experiments.

This new approach opens the possibility of assimilating and making use of all the stored information when one solves the problems of long and extra-long range forecasting. Moreover, there are some possibilities for decreasing the influence of errors in the initial fields and some systematic errors in numerical models using this approach.

In order to use (38) for the purposes of long-range forecasting, it is necessary to develop some technique for the assessment of the deviations of the energy sources and sinks from their climatic values over the forecast time interval. Probably, the hopes here may be connected with the use of some statistical methods and satellite information on the World Ocean and planetary cloudiness.

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