THE FLUX-CORRECTION METHOD

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1. INTRODUCTION

The flux-correction method introduced by Boris (1971, 1975a,b) is a technique for obtaining an improved numerical solution of the mass continuity equation

$$\dot{\rho} = - \nabla.(\rho \underline{V}) \qquad (1)$$

in the vicinity of steep gradients.

Besides the usual requirement of stability and consistency, the numerical scheme is subject to the conditions:

A. Positive solutions should result from positive initial values.

B. For $\underline{V} = 0$ the numerical process should not change ρ.

Condition B is rather important for meteorological applications. It is generally not satisfied for flux-corrected methods, which tend to retain some smoothing for $\underline{V} = 0$.

Schemes subject to condition B are called phoenical and represent the most interesting branch of flux-corrected methods for meteorological use.

A possible meteorological application of this technique is for the advection of moisture in the vertical. Some of the errors involved in moisture advection can be seen using the square wave test. This test uses a one-dimensional version of (1) with periodic boundary conditions and constant wind V; the initial condition is a step function. Here we are particularly interested in advection over the whole vertical extent of the atmosphere, which we assume to be represented by about 20 layers. We use $V\frac{\Delta t}{\Delta x} = .2$, so that in 5 timesteps a disturbance moves one grid-interval. Fig.1 gives the results of the centered difference scheme for 50, 100, and 600 timesteps. Even for advection over short distances, considerable errors arise; this leads to negative field values, which are physically unrealistic for a moisture field.

Numerical models usually include some simple remedy of this problem. One method is to collect positive moisture from surrounding gridpoints whenever a gridpoint value becomes negative. Of course this collection must be done in a mass conserving way.

Fig.2 shows a forecast using this procedure. For short periods there is a definite improvement; however for longer periods (N=600, Fig.2c) it is apparent that the accuracy of this second order method is still insufficient. Particularly disturbing is the appearance of artificial plateaus.

The results shown in Fig.2a,b are used to define the level of performance which should be improved upon by the flux-corrected methods.

The introduction of highly accurate methods does not in itself improve the forecast of steep gradients. Fig.3a,b gives the example of the spectral method in which negative moisture has been avoided by using the same procedure as applied in Fig.2a,b. We see that there are heavy oscillations at the upper shoulder of the steep gradient region. Also note that in this example the original Gibbs-error is noticeably enhanced by the time truncation error introduced by the leapfrog-scheme. This kind of performance is not surprising since an increase in the order of the numerical method does not necessarily give an increase in the accuracy with which the smallest scales are treated.

The aim of this paper is to review the flux-correction technique with respect to the vertical discretization of the moisture field. However we will not be

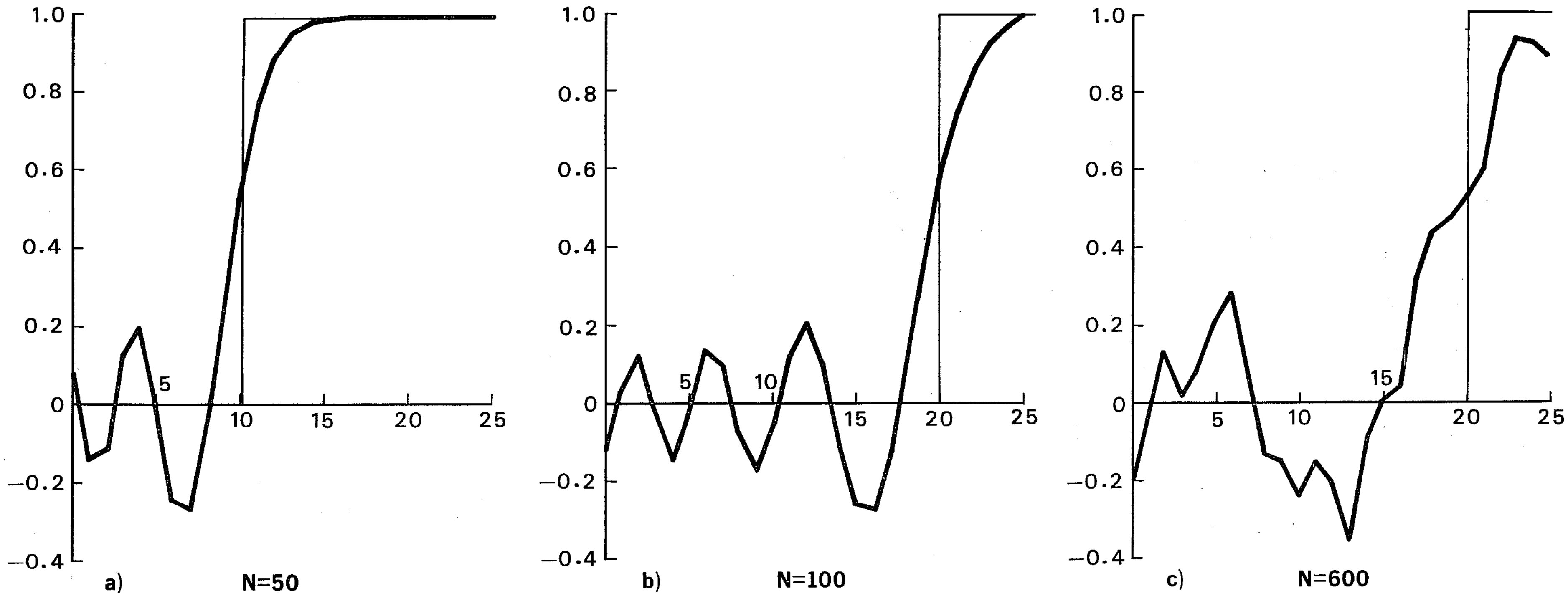

Fig. 1 Square wave test for the leapfrog centered difference scheme, $\frac{V\Delta t}{\Delta x} = .2$, forecast fields for different numbers N of time steps. Thin line = exact solution.

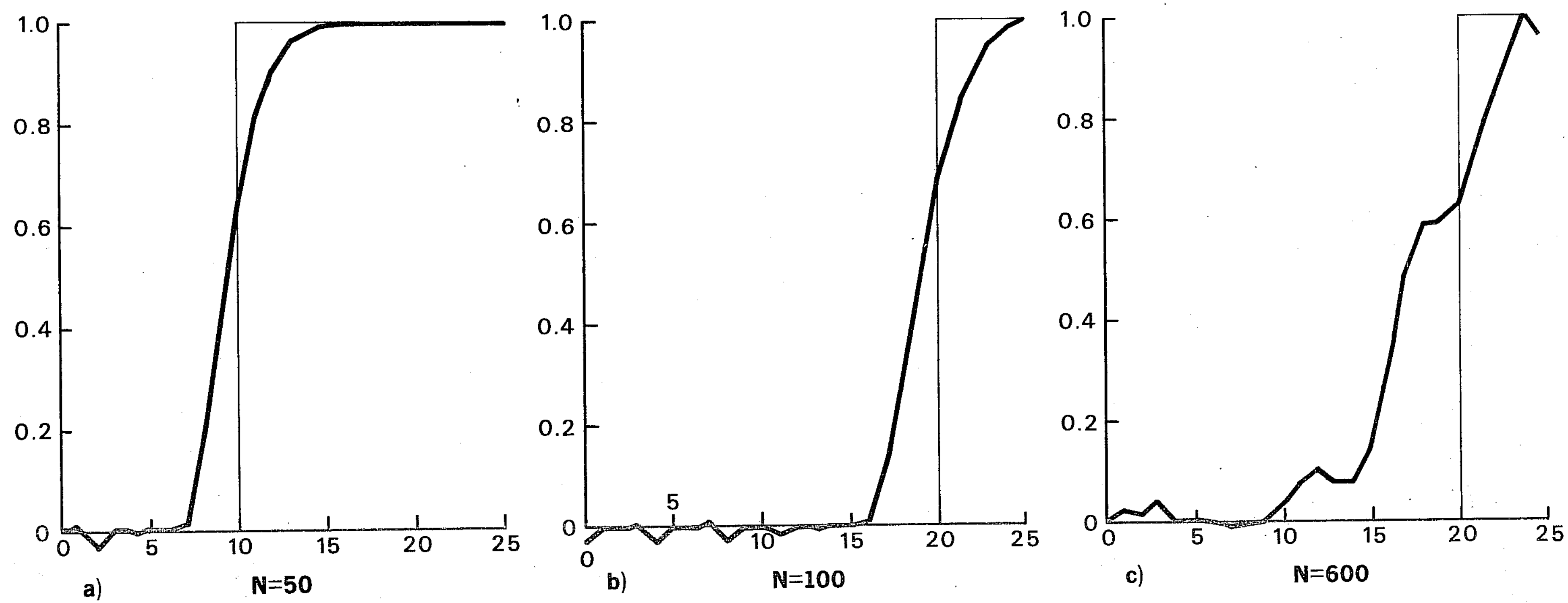

Fig. 2 As Fig. 1, but for the leapfrog centered differences scheme with collection of negative moisture.

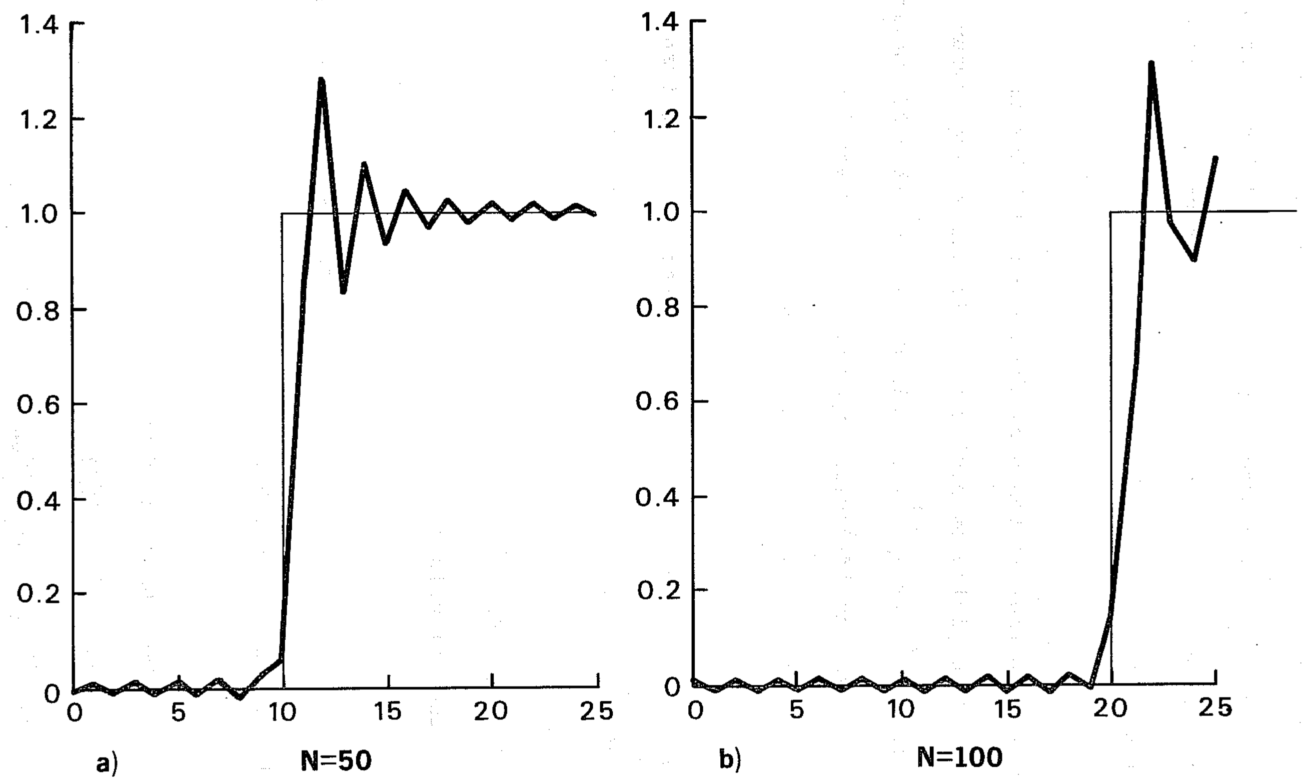

Fig. 3 As Fig. 1a, b, but for spectral method with collection.

concerned with the further development of the method for the discretization of the two-dimensional Navier-Stokes equations; this is given in Boris (1972).

2. THE BASIC FORMALISM OF THE FLUX-CORRECTION METHOD

The flux-correction method first introduced by Boris (1971) as a rather special scheme, was later developed by Boris (1975a,b) into a rather flexible technique, which makes it possible to derive flux corrected versions of most of the usual finite difference schemes.

The flux-correction method is based on the notion that the usual numerical schemes can be made positive by adding a sufficiently large amount of diffusion. (A scheme $\{\rho_\nu^t\} \rightarrow \{\rho_\nu^{t+\Delta t}\}$ is called positive if $\rho_\nu^t \geqslant 0$ implies $\rho_\nu^{t+\Delta t} \geqslant 0$). A nonlinear antidiffusion operator is then applied which removes the diffusion in regions where it is not needed.

The flux correction thus consists of three operations: the transport (denoted by T) the diffusion (denoted by D) and the antidiffusion operator (denoted by A). It can be written as:

$$\{\rho_\nu^1\} = (1+A)\ (1+T+D)\{\rho_\nu^o\} \tag{2}$$

where ν denotes the gridpoints.

The diffusion can be supplied by an explicit diffusion term or by an implicit diffusion of the transport operator.

Let the operator (1+T+D) have the form

$$\tilde{\rho}_\nu(t+\Delta t) = \rho_\nu(t)+\Delta t \quad \dot{\rho}_j(t+\Delta t/2)$$
$$+ \{\eta_{\nu+1/2}\ [\rho_{\nu+1}(t)) - \rho_\nu(t)] - \eta_{\nu-1/2}\ [\rho_\nu(t) - \rho_{\nu-1}(t)]\} \tag{3}$$

with

$\eta_{\nu-1/2}\ (\rho_\nu(t) - \rho_{\nu-1}(t))$ being the flux associated with the operator D.

To define the antidiffusion step, we require the provisional flux

$$\tilde{\phi}_{\nu+1/2} = \eta_{\nu+1/2}\ [\rho_{\nu+1}(t+\Delta t) - \rho_{\nu}(t+\Delta t)] \qquad (4)$$

The corrected flux is computed according to the following expression which is the central equation of the flux correction technique.

$$\phi_{\nu+1/2} = S \max\left\{0, \min\left(\tilde{\Delta}_{\nu-1/2}\cdot S, \left|\tilde{\phi}_{\nu+1/2}\right|, \tilde{\Delta}_{\nu+3/2}\cdot S\right)\right\} \qquad (5)$$

with

$$\tilde{\Delta}_{\nu+1/2} = \tilde{\rho}_{\nu+1}(t+\Delta t) - \tilde{\rho}_{\nu}(t+\Delta t)$$

and $S = \operatorname{sgn} \tilde{\Delta}_{\nu+1/2}$

The field at t+δt is then obtained by applying the corrected antidiffusion:

$$\rho_{\nu}(t+\Delta t) = \tilde{\rho}_{\nu}(t+\Delta t) - \phi_{\nu+1/2} + \phi_{\nu-1/2} \qquad (6)$$

Detailed descriptions of many versions of the flux-corrected schemes (like Lax-Wendroff, upstream differencing, and leapfrog) are given in Boris (1975a,b). However some of these schemes, like the leapfrog scheme go slightly beyond the formalism defined by Eqns.(2)-(6). Also, the flux-corrected Fourier scheme can not be easily generalised for application to nonlinear problems. Therefore its results were derived only to get an impression of the performance of a highly developed flux corrected scheme.

Here we do not give details of how to implement all the schemes mentioned above; for this information refer to the papers cited. However, as an example of the implementation of a flux-corrected scheme, we will describe the scheme SHASTA introduced by Boris (1971).

3. SHASTA, AN EXAMPLE OF A FLUX-CORRECTED SCHEME

The characteristic feature of this scheme is the use of a quasi-Langrangian scheme for the transport stage. It is described for a constant gridlength Δx.

We assume the field ρ is composed of trapeziums, as shown in Fig. 4 (solid lines). For constant Δx this assumption is consistent with the mass formula

$$M_o = \sum_{\nu} \rho_{\nu} \, \Delta x \tag{7}$$

The chain-dashed trapeziums result from the application of a Langrangian transport step and the small arrows indicate the Langrangian advection of the gridpoints

$$\delta_{\nu} = V_{\nu} \, \Delta t \tag{8}$$

We assume

$$\max_{\nu} \left| \frac{V_{\nu} \, \Delta t}{\Delta x} \right| < \frac{1}{2} \tag{9}$$

in order to ensure that no crossing of gridpoints occurs.

The amplitude values of $\rho^{\pm}$, which belong to the advected trapezium, are

$$\rho_{\nu}^{+} = \frac{\rho_{\nu+1}^{0} \, \Delta x}{\Delta x + \Delta t (V_{\nu+1}^{1/2} - V_{\nu}^{1/2})} \tag{10}$$

$$\rho_{\nu}^{-} = \frac{\rho_{j}^{0} \, \Delta x}{\Delta x + \Delta t (V_{\nu+1}^{1/2} - V_{\nu}^{1/2})}$$

Where $V_{\nu}^{1/2}$ is the V_{ν} value at $t + \Delta t/2$.

The diffusion step is performed by transforming the shaded area F_{ν} in Fig.4 into a square, then

$$\tilde{\rho}_{\nu}(t+\Delta t) = \frac{F_{\nu}}{\Delta x} \tag{11}$$

For a uniform velocity field, the time-translation equations are:

$$\tilde{\rho}_{\nu}^{n+1} = \rho_{\nu}^{n} - \frac{V\Delta t}{2\Delta x} (\rho_{\nu+1}^{n} - \rho_{\nu-1}^{n}) + \left(\frac{1}{8} + \frac{1}{2} \left(\frac{V\Delta t}{\Delta x}\right)^{2}\right)(\rho_{\nu+1}^{n} - 2\rho_{\nu}^{n} + \rho_{\nu-1}^{n}) \tag{12}$$

Where the index n denotes the time level.

Therefore, in the linear case the scheme reduces to the one-step Lax-Wendroff scheme.

From (11) or (12) we see that the diffusion is still large even when V=0. Fig.5 shows the field after one timestep with V=0.

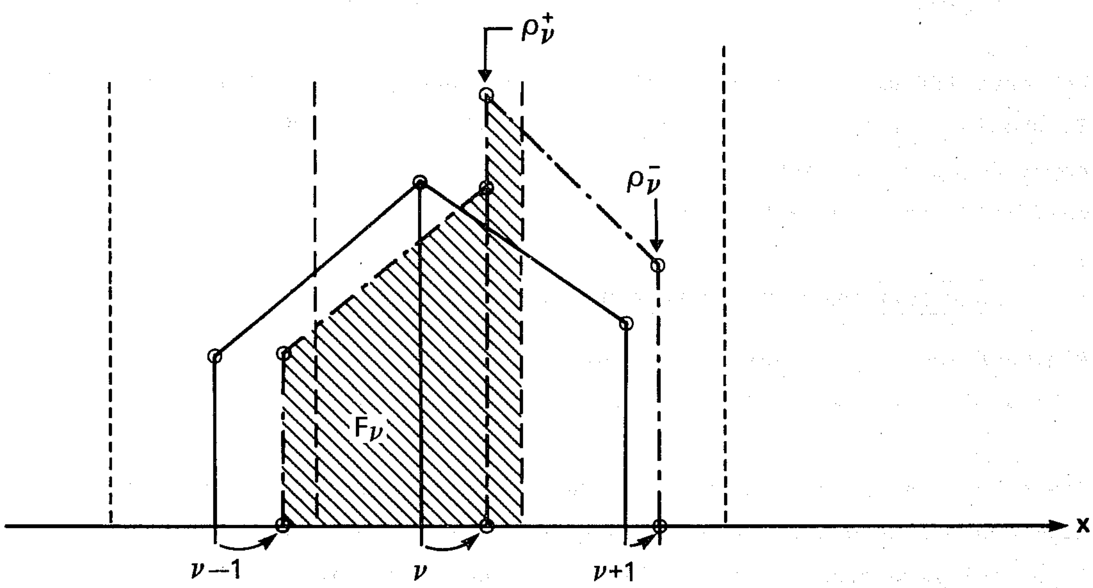

Fig. 4 ———— Field at time t. —·—·— Field at time t + Δ t before the diffusion step. The staded area F_ν is used to compute $\rho(t+\Delta t)$ from $\rho_\nu(t+\Delta t)$. $\Delta x = F_\nu$. The small arrows represent the lagrangian advection of the gridpoints in one time step.

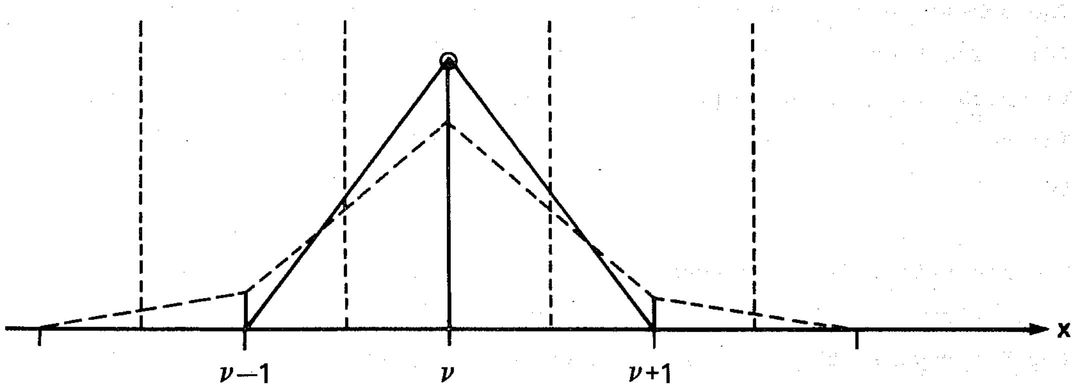

Fig. 5 Action of the numerical diffusion for zero velocity.

———————— = initial

- - - - - - - - - = after one time step.

The antidiffusion step is performed according to (4) and (5) using

$$\eta_{\nu+1/2} = \frac{1}{8} \tag{13}$$

The antidiffusion removes the effect of the smoothing for the larger scales. It should be noted, however, that the smoothing of the field shown in Fig.5 is not removed by the antidiffusion step, so that the dashed line also indicates the field after the antidiffusion step.

4. NUMERICAL TESTS FOR SOME FLUX CORRECTED METHODS

Fig.6a-f shows the result of the square wave test, taken from Boris (1972, 1975,a,b), for several flux-corrected methods.

The schemes shown are the SHASTA-scheme described in Sect.3, the phoenical-Lax-Wendroff scheme and the leapfrog 1-scheme of Boris (1975a), along with the Donor cell, reversible flux-corrected transport scheme and the Fourier method described by Boris (1975b).

The test uses the linear advection equation ((1) with V=const) with $\frac{V\Delta t}{\Delta x} = .2$ for 800 cyles.

Figs.6a-d show the result of explicit flux corrected schemes, based on (2) to (6), except for slight modifications of the leapfrog 1-scheme. The phoenical Lax-Wendroff scheme appears to be the best of these schemes, since it requires 7 points on the shoulder.

The schemes presented in Fig.6e,f go essentially beyond the formalism of (2)-(6). The Fourier method (Fig.6f) uses knowledge of the exact solution for V=constant, and it's application to the general case is not straightforward. The scheme was analysed in order to get a limit on the behaviour of rather good schemes.

The reversible flux-corrected transport method, described in Boris (1975b), is an implicit method requiring the inversion of a tridiagonal matrix. Boris (1975b) reports that this scheme behaves badly in nonlinear problems. For linear advection, however, the scheme shows the best results of all schemes which can be generalised for the nonlinear case.

There are typical errors associated with the flux-correction method. The phoenical schemes remove the diffusion for V=0 for only the larger scales. It

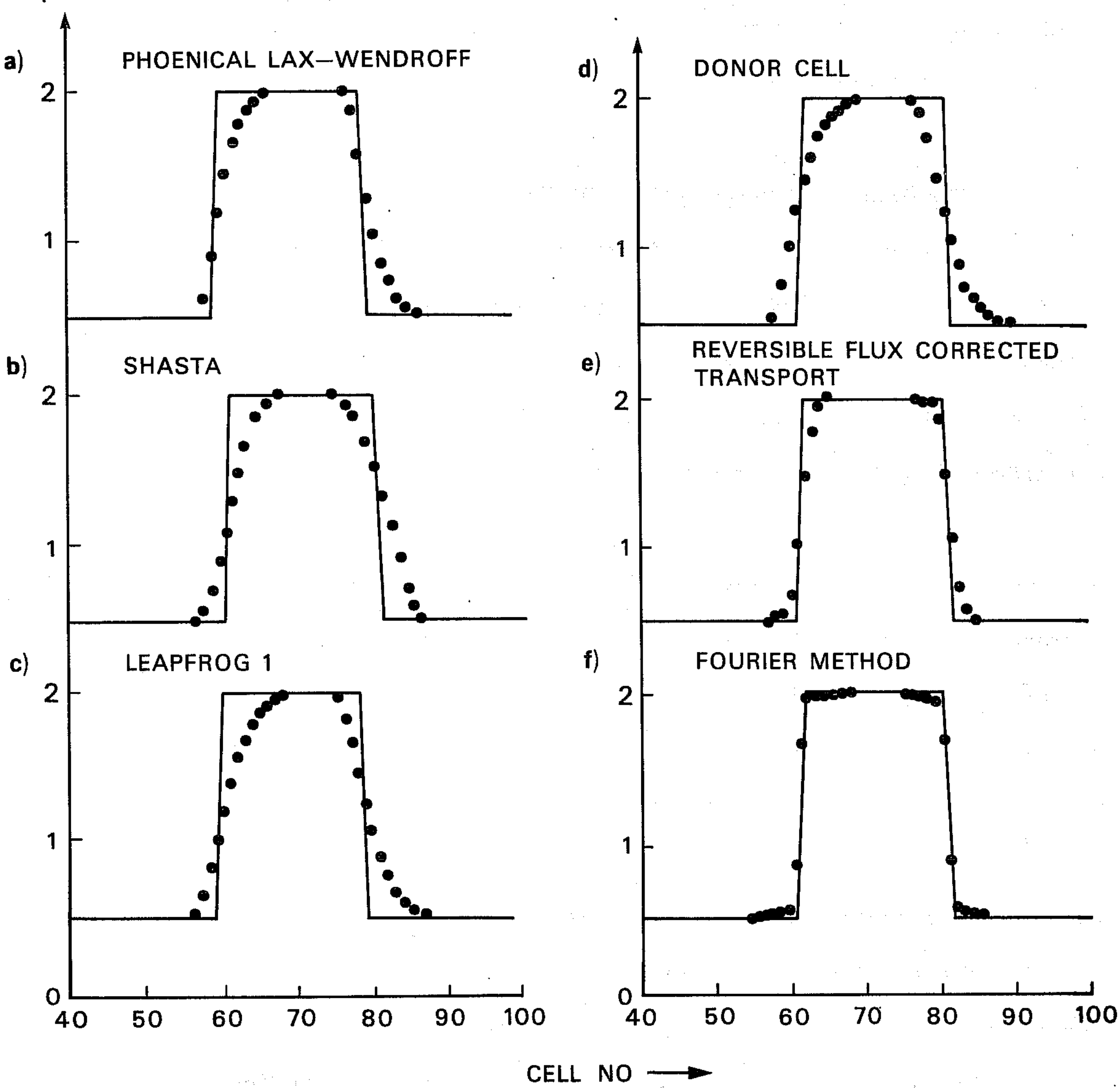

Fig. 6 The square wave test for several flux corrected schemes, according to Boris (1975 a,b) for 800 cycles with $\frac{V\Delta t}{\Delta x} = .2$

——————— = exact solution

• • • • • • • = approximate solution

is found that sharp peaks in the fields are transformed into typical plateaus. This error, which is called clipping, is illustrated in Fig.7. It shows the result of one timestep with V=0 on a sharp peak. In particular the SHASTA scheme of Sect.3 smears the peak considerably.

5. TWO EXAMPLES OF FLUX-CORRECTED DIFFUSION

Instead of applying diffusion and antidiffusion, it is natural to try to use the diffusion in a controlled way. This method, called flux corrected diffusion, was tested by Boris (1975a); the results compared unfavourably with the SHASTA method and this reflects the fact that the diffusion and antidiffusion can improve the phase error of a method.

Here we try a refined version of this method based on the fact that the Lagrangian advection step, being the first stage of the SHASTA scheme, is a rather good approximation of the dynamic equations. However, the remapping of the resulting field onto the original grid is a rather coarse approximation, making the antidiffusion steps necessary to partly reverse it.

We now try to replace the latter operation by a more refined one, so that the use of the antidiffusion operation can be avoided. We are also looking for schemes which, in the V=0 case, leave the field unchanged and have no clipping error. This is achieved by computing the difference of the Lagrangian advected field at $t+\Delta t$ and the field at time t and remapping this field to the original grid, whilst observing the positivity condition.

The first stage of the SHASTA formalism can be considered as a first order finite element Langrangian method. Here we will also consider the second order element representation. The field representation are then

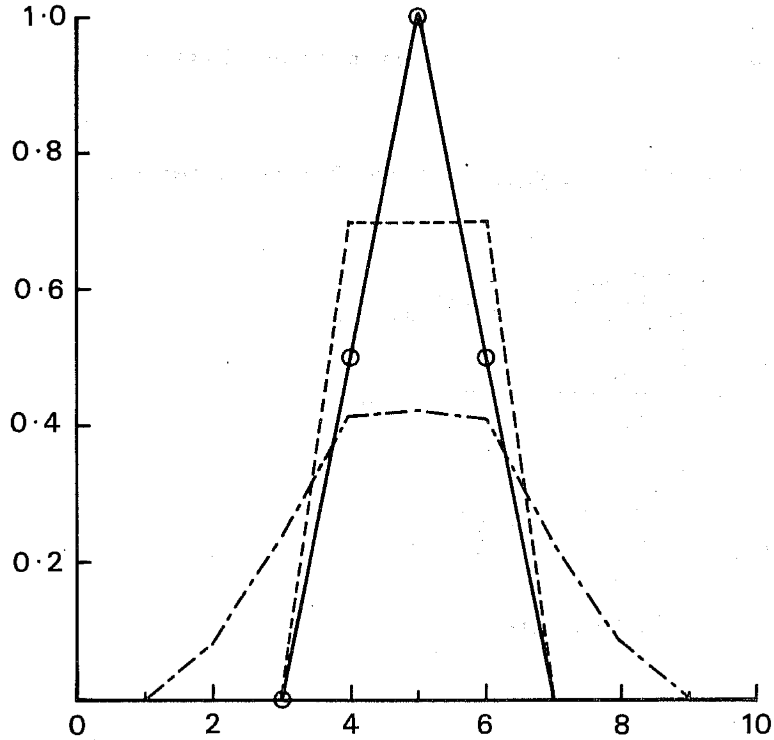

Fig. 7 The clipping error according to Boris 1975. Result of one timestep with V=0
——— = initial
------- = phoenicial solution
—·—·— = SHASTA scheme of section 3.

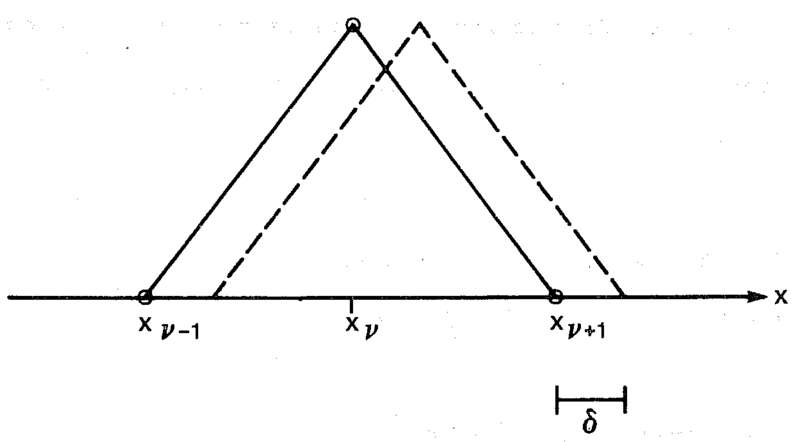

Fig. 8 The definition of e'
——— = e
– – – – = e'

$$\rho(x) = \sum_\nu \rho_\nu e_\nu(x) \qquad \text{[first order elements]} \tag{14}$$

$$\rho(x) = \sum_\nu \left\{\rho_\nu e_\nu + \rho_{\nu+1/2}\, e_{\nu+1/2}\right\} \qquad \text{[second order elements]} \tag{15}$$

with

$$e_\nu(x) = \begin{cases} \dfrac{x-x_{\nu-1}}{x_\nu - x_{\nu-1}} & \text{for } x\varepsilon\{x_{\nu-1}, x_\nu\} \\ \dfrac{x_{\nu+1}-x}{x_{\nu+1}-x_\nu} & \text{for } x\varepsilon\{x_\nu, x_{\nu+1}\} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$e_{\nu+1/2}(x) = \begin{cases} \dfrac{4(x-x_\nu)(x_{\nu+1}-x)}{(x_{\nu+1} - x_\nu)^2} & \text{for } x\varepsilon\{x_\nu, x_{\nu+1}\} \\ 0 & \text{otherwise} \end{cases}$$

It is not self-evident that an increase in the order of the field representation will improve the behaviour of the scheme in the vicinity of steep gradients, since an increased order necessarily increases only the accuracy for the larger scales. However the numerical experiments have shown an improved performance when the order of the field representation is increased from 0 to 2.

An impression of these flux-corrected diffusion schemes can be obtained by considering the first order case for constant velocity and constant grid resolution.

The Lagrangian step can then be written as

$$\rho(x) = \sum_\nu \rho_\nu\, e_\nu(x) + \rho'(x) = \sum_\nu \rho_\nu\, e'_\nu(x) \tag{17}$$

with

$$e'_\nu(x) = e_\nu(x-\delta) \text{ where } \delta = V\Delta t$$

The definition of $e'(x)$ is illustrated in Fig.8.

Define

$$a_\nu = \int_{x_{\nu-3/2}}^{x_{\nu-1/2}} e'_\nu(x) - e_\nu(x)\ dx$$

$$b_\nu = \int_{x_{\nu-1/2}}^{x_{\nu+1/2}} e'_\nu(x) - e_\nu(x)\ dx \qquad (18)$$

$$c_\nu = \int_{x_{\nu+1/2}}^{x_{\nu+3/2}} e'_\nu(x) - e_\nu(x)\ dx$$

We compute the mass entrainment of the interval $(x_{\nu-1/2}, x_{\nu+1/2})$ as

$$\tilde{\alpha}_\nu = \rho_{\nu+1} a_{\nu+1} + \rho_\nu\ b_\nu + \rho_{\nu-1}\ c_{\nu-1} \qquad (19)$$

The simple collection procedure described in the introduction is used to derive α_ν in a way which is consistent with the positivity condition. The time-stepping is then done by using

$$\rho_\nu\ (t+\Delta t) = \rho_\nu(t) + \alpha_\nu/\Delta x \qquad (20)$$

It should be noted that the second order elements require a more refined kind of mass redistribution.

The result of the square wave test (using the same conditions as for Figs.1-3) is shown in Fig.9. The quality of simulation for extensive advection (N=600) is similar to that of the flux corrected leapfrog 1 method or the Donor cell method. However the result is much better than the flux-corrected diffusion scheme of Boris (1975a). The more refined method of diffusion correction results in no loss of accuracy for performing diffusion and antidiffusion in one step.

The result for the second order elements are shown in Fig.10c. The predictions for N=600 are now comparable to the phoenical-Lax-Wendroff scheme, which represents one of the best explicit schemes presented in Boris (1972, 1975a,b).

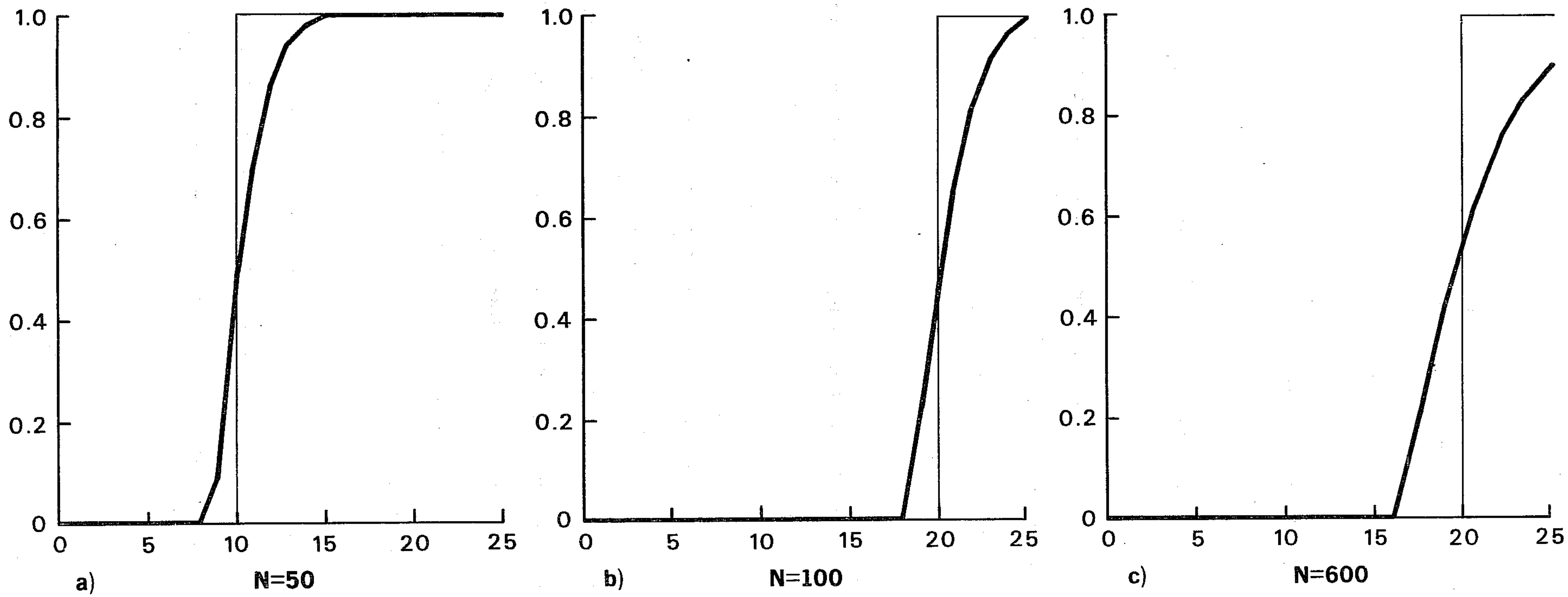

Fig. 9 The square-wave test for the Lagrangian first order finite element diffusion corrected scheme, $\frac{V \Delta t}{\Delta x} = .2$, N is the number of time steps.

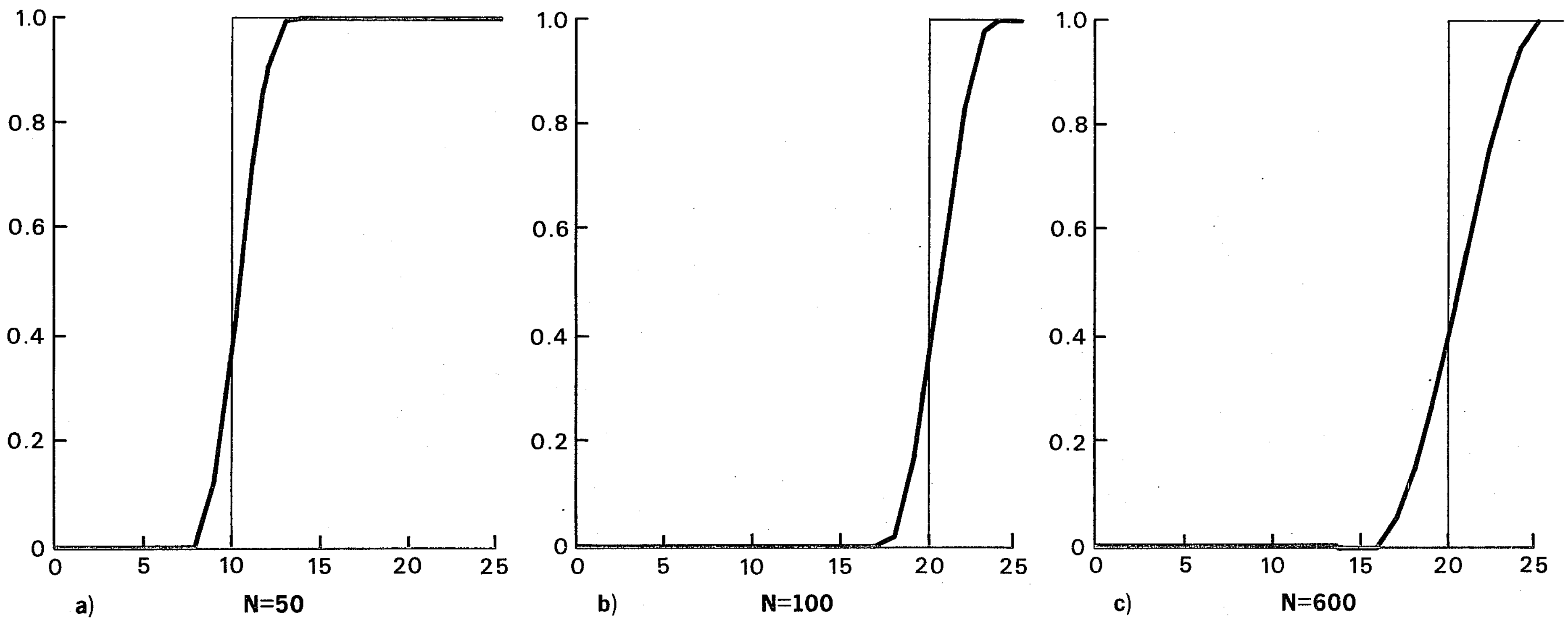

Fig. 10 As Fig 9, but for second order elements.

Figs.9 and 10a,b show the advection over 10 and 20 gridpoints, which approximately represent the number of levels used in the ECMWF model.

6. CONCLUSIONS

The square wave test shows that there is some potential for improvement by introducing the flux-corrected transport method in connection with the simple finite difference schemes now used in the ECMWF model. When envisaging the transition to some highly accurate method of vertical discretization, the possibility of increasing the accuracy by diffusing and selectively antidiffusing becomes less interesting; therefore some simplified approach to flux-correction might be worth trying.

The schemes which are feasible still produce a considerable amount of smearing of steep gradients for extended forecasts. Consequently further research concerning these schemes may be necessary. One obvious approach is to solve the technical problems involved in the implementation of the high performance flux-corrected methods, such as the reversible flux corrected method or the Fourier method. However it should be noticed that the use of the Fourier method for vertical discretization poses some problems even without flux-correction. Another approach could be based on the fact that the Lagrangian forecast step is quite accurate for large timesteps. If one treats the vertical advection term with an increased timestep, this would automatically reduce the number of grid-remapping steps and thus increase the accuracy. This approach was used for a more general equation by Leveque (1982), who found a dramatic increase of accuracy associated with this method.

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