NUMERICAL METHODS FOR NONHYDROSTATIC MODELS

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1. INTRODUCTION

The numerical modelling of nonhydrostatic atmospheric flow possesses a history similar to that of modelling hydrostatic flow. The pioneering models in both disciplines are based on approximate equations that filter modes that are computationally difficult to include. Examples of these are the barotropic models used in the earliest NWP work and the simplest anelastic models used in the earliest studies of convection. As time has passed more complete equation sets have been used in the study of both hydrostatic and nonhydrostatic flow. Today, while some models based on approximate equation sets remain appealing for certain applications, many nonhydrostatic models now solve the fully compressible Navier-Stokes equations with, of course, turbulence, microphysics and radiation still largely parameterized. In this paper, numerical methods used to solve the dynamical equations describing nonhydrostatic motion are reviewed.

The earliest nonhydrostatic models filtered sound waves by integrating equations based on the anelastic approximation of Ogura and Phillips (1962). Acoustic modes can exist in solutions to the full set, and while sound waves are not meteorologically significant (they possess little energy), their rapid propagation can impose severe limits on the time steps used in models. The anelastic approximation and several related variants, along with coordinate system and variables used in nonhydrostatic models will be discussed in Section 2.

The first numerical integrations of the fully compressible equations were performed by Hill (1974) using an explicit timestep limited by the acoustic modes. However, explicit integrations of the fully compressible set are prohibitively expensive. Two approaches are used to reduce the cost of the simulations. Tapp and White (1976) integrated the compressible system with a semi-implicit scheme; terms responsible for the acoustic modes are integrated implicitly. Klemp and Wilhelmson (1978) introduced a scheme whereby the terms responsible for the acoustic modes are integrated with an explicit, smaller timestep separate from the other terms. This approach is economical because the acoustic-mode terms are inexpensively evaluated compared with the nonlinear terms, microphysics, etc... These methods for solving the fully

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compressible system, along with their more recent refinements, will be reviewed in Section 3. Numerical methods for solving the implicit relations arising from semi-implicit solution techniques for the fully compressible systems and for solving the elliptic equations that arise in the anelastic systems will be considered in Section 4.

Recently, several nonhydrostatic models have been used to perform large-scale flow simulations. The feasibility of performing large-scale flow simulations with nonhydrostatic models is demonstrated in integrations using the Clark (1977) anelastic model by Polavarapu (1990), with the Klemp-Wilhelmson model by Snyder *et al*, (1991), with the latest versions of the British Meteorological Office model (Cullen 1990) and with the semi-implicit, semi-Lagrangian model of Tanguay *et al* (1990). In many cases the integrations are comparable in cost with the hydrostatic models; both gravity and acoustic waves can be advanced in a manner such that they do not restrict the model timestep. Moreover, the techniques used to advance the acoustic and gravity waves are often much simpler than those used to economically integrate gravity waves in hydrostatic models. We comment further on these considerations in the following sections.

2. EQUATIONS

2.1 Fully Compressible Equations

In atmospheric modelling, we ultimately wish to solve the fully compressible Navier-Stokes equations. For the purposes of this paper, we neglect the Coriolis terms and detailed presentations of microphysics and mixing parameterizations; they have little effect on the general structure of the numerical schemes. With these simplifications the governing equations can be stated as

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k} + D \tag{1}$$

$$c_v\frac{dT}{dt} = -RT\nabla\cdot\mathbf{V} + D_T + S_T \tag{2}$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{V} \tag{3}$$

$$p = \rho RT \tag{4}$$

These are the momentum, thermodynamic, mass conservation and state equations respectively for the standard variables, where $\mathbf{V} = (u, v, w)$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

and D and S represent dissipation and sources. Most nonhydrostatic models solve equations posed in an (x,y,z) coordinate system, although to include terrain the vertical coordinate is often transformed. We consider the transformations in Section 2.3.

Few models solve the $\rho-T$ system (1)-(4). Three major modifications to the set are commonly employed. The first is to replace (2) with an thermodynamic equation posed in terms of the potential temperature θ, where

$$\theta = T\left(\frac{p_o}{p}\right)^{R/c_p}.$$

The new thermodynamic equation is

$$\frac{d\theta}{dt} = D_\theta + S_\theta \tag{5}$$

and the new equation of state (4) becomes

$$p = (p_o^{-R/c_p}\rho R\theta)^{c_p/c_v}. \tag{6}$$

The advantage of using the potential temperature θ over the temperature T is that the thermodynamic equation is simplified. Indeed, in an adiabatic atmosphere it reduces to $d\theta/dt = 0$. The system (1), (3), (5) and (6), which we denote as the $\rho - \theta$ system, is solved in a model described by Tripoli and Cotton (1982) where the equations are posed in terms of variables that are perturbations about a hydrostatic, time-invariant mean state. Most nonhydrostatic models use θ as the thermodynamic variable.

The second major modification, typically used when θ is the thermodynamic variable, is to replace the pressure with the Exner function

$$\Pi = c_p\left(\frac{p}{p_o}\right)^{R/c_p}.$$

With this substitution the momentum equation (1) becomes

$$\frac{d\mathbf{V}}{dt} = -\theta\nabla\Pi - g\mathbf{k} + D \tag{7}$$

and the new equation of state is

$$\Pi = (Rp_o^{-1}\rho\theta)^{R/c_v}. \tag{8}$$

Yet another variant is that used by Tanguay *et al* (1990), where pressure is replaced by $q = \ln(p/p_o)$. The pressure gradient term in the momentum equation (1), $\rho^{-1}\nabla p$, is replaced by $RT\nabla q$.

The third major modification to the original equations is to recast the continuity equation in terms of Π, p or q. This can be accomplished in a variety of ways, the most straightforward being to take the material derivative of the equation of state and to use the continuity equation to eliminate ρ. The new continuity equation for this $\Pi - \theta$ system is

$$\frac{d\Pi}{dt} = -\frac{c_s^2}{c_p\theta}\nabla \cdot \mathrm{V} + \frac{c_s^2}{c_p\theta^2}\frac{d\theta}{dt} \tag{9}$$

In this system no equation of state is needed; (5), (7) and (9) form a closed system. This $\Pi - \theta$ set is used in several numerical models that are discussed in the next section, including models developed by Tapp and White (1976), Klemp and Wilhelmson (1978), and others. Alternatively, the continuity equation can be expressed in terms of the pressure p as

$$\frac{dp}{dt} = -c_s^2\rho\nabla \cdot \mathrm{V} + \frac{c_p}{c_v}\frac{p}{\theta}\frac{d\theta}{dt} \qquad (10).$$

This $p - \theta$ system is used in some nonhydrostatic models, such as the model of Cotton and Tripoli (1978). Note that an equation of state is still needed in this system. Finally, the continuity equation for the $q - T$ system is

$$\frac{c_v}{c_p}\frac{dq}{dt} = -\nabla \cdot \mathrm{V} + (D_T + S_T)/T \tag{11}$$

No equation of state is needed here.

2.2 Approximations to the Continuous Equations

In general, one can partition the existing models into those that solve the fully compressible equations and those that solve the anelastic equations. In this section we discuss the anelastic approximation first, and then consider some approximations used in the fully compressible models.

The earliest nonhydrostatic models did not solve the fully compressible equations because the presence of acoustic modes severely limited the model timesteps. Ogura and Phillips (1962) introduced the anelastic approximation that results in replacing the continuity equation (3) with

$$\nabla \cdot \rho_o \mathrm{V} = 0. \tag{12}$$

This removes sound waves from the system. The rigorous scaling analysis which underlies this approximation requires that the potential temperature θ_1 have only small variations from a reference temperature θ_o. For dry convection the approximation appears sound, however, the assumption of near constant θ may not be valid when phenomena such as deep, moist convection or stratospheric gravity waves are being simulated (Durran, 1989). For the remainder of this paper basic state mean quantities will have a subscript 0 and perturbations a subscript of 1.

Several alternative anelastic systems have been put forth that address some of the problems noted by Durran. The anelastic approximation of Wilhelmson and Ogura (1972) uses a non-isentropic base state. However, a closed form energy integral cannot be derived for it without further approximation whereas for Ogura and Phillips' set a closed form exists. Lipps and Hemler (1982) present an anelastic approximation wherein the necessary assumption is that the base state θ be a slowly varying function of z. This set has a closed form energy integral. A more recent analysis of this set (Lipps, 1990) lends further evidence to its validity and applicability.

The anelastic approximations we have discussed all use the continuity equation (12) with differing definitions of the base state and slightly different truncations of the momentum equations. Durran (1989) has introduced another anelastic approximation which leads to a continuity equation of the form

$$\nabla \cdot (\rho_o \theta_o \mathrm{V}) = \frac{H}{c_p \Pi_o} \tag{13}$$

where H is the rate of heating per unit volume. The assumptions made are that the Lagrangian time-scale associated with a disturbance is much larger than the sound-wave time scale and that $\Pi_1 \ll \Pi_o$. This approximate set conserves an energy form that is similar to that conserved by the fully compressible set and the energy conservation is achieved independent of the mean state stratification.

With the introduction of an anelastic approximation, a prognostic equation for the pressure (or density) no longer exists. A diagnostic equation for the pressure is produced by forming a divergence equation from the momentum equations. The Helmholtz equation used by Ogura and Phillips is

$$\nabla \cdot \rho_o \nabla \Pi_1 = \mathrm{RHS}. \tag{14}$$

While ρ_o might not vary horizontally but does vary vertically. Wilhelmson and Ogura derive a similar equation using their approximation with $\rho_o\theta_o$ replacing ρ_o; Lipps and Hemler solve a similar equation. Using (13), Durran derives a diagnostic equation for Π_1 of the form

$$\nabla \cdot \rho_o\theta_o\theta\nabla\Pi_1 = \text{RHS}.$$

This system is slightly more complex than (14). The new equation no longer has constant coefficients in the case where the mean state varies only in z. We will discuss solution procedures for these elliptic equations in Section 4.

Most nonhydrostatic models solve equations for variables that are perturbations about a hydrostatic, time-invariant mean state. This approach reduces truncation and roundoff errors in calculations of the hydrostatic balance. Also, certain terms are linearized about the mean state in several models. These relatively minor approximations will not be discussed. For cloud-scale simulations (L < 100km), linearization about a vertically varying mean state is reasonable. However, for larger-scale flows the mean state may be difficult to define, hence linearization or any approximation based on a mean state may lead to significant errors.

Two approximations often used in fully compressible models deserve comment. First, certain terms in the pressure equations (9), (10) or (11) that replace the original continuity equation (3) are often discarded. Equation (9) can be written as

$$\frac{\partial\Pi}{\partial t} + \frac{c_s^2}{c_p\theta}\nabla \cdot \mathbf{V} + w\frac{\partial\Pi}{\partial z} = -\mathbf{V}_H \cdot \nabla\Pi + \frac{c_s^2}{c_p\theta^2}\frac{d\theta}{dt} \tag{15}$$

Klemp and Wilhelmson state that the terms on the right in (15) have little influence on the cloud-scale, and simulations with and without these terms differ little. Second, the sound speed $c_s(x,y,z,t)$ is often replaced by a single constant sound speed. Furthermore, in some models the sound speed used is significantly less than the actual sound speed, often only 50 to 75% of the actual value. This *quasi-compressible* approximation is discussed by Droegemeier and Wilhelmson (1987).

2.3 Alternate Vertical-Coordinate Systems

Several models use vertical coordinate systems other than the geometric height z. The inclusion of terrain in a model is usually brought about by using a transformed-z vertical coordinate. The most popular transformation is that given by Gal-Chen and Somerville (1975)

$$\tilde{z} = H\frac{z - z_s}{H - z_s},$$

where z_s is the height of the surface and H is the top of the domain. Various other transformed-z coordinates are possible. For example, Carpenter (1979) introduces terrain into the semi-implicit, fully compressible model of Tapp and White using the independent variable

$$\eta = z - H$$

These transformations introduce only minor changes to the governing equations.

Various forms of a pressure coordinate system have been used. Here we consider both a pressure-coordinate system and two systems based on a σ-coordinate, where

$$\sigma = \frac{p - p_{top}}{p^*}. \quad (16)$$

In one the pressure p^* is the surface pressure and in the other p^* is a fixed reference-state pressure that depends only on terrain altitude.

Dudhia (1991) has developed a nonhydrostatic model using a numerical scheme similar to that in Klemp and Wilhelmson (1978) except that he replaces Klemp and Wilhelmsons' vertical coordinate z with the σ-coordinate system (16) where p^* is a fixed reference-state pressure. The reference state satisfies the hydrostatic equation $\partial p_o/\partial z = -\rho_o z$; $p_o(z)$ is defined for any terrain height and $p^* = p_o(z_{surface})$. In the new vertical-coordinate system the horizontal and vertical momentum equations are

$$\frac{du}{dt} = -\frac{1}{\rho}(\frac{\partial p_1}{\partial x} - \frac{\sigma}{p^*}\frac{\partial p^*}{\partial x}\frac{\partial p_1}{\partial \sigma}) + D_u$$

and

$$\frac{dw}{dt} = \frac{\rho_o}{\rho}\frac{g}{p^*}\frac{\partial p_1}{\partial \sigma} - \frac{gc_v}{c_p}\frac{p_1}{p} + g\frac{p_o}{p}\frac{T_1}{T_o} - \frac{gR}{c_p}\frac{p_1}{p} + D_w,$$

where the variables have been expanded into mean and perturbation components. Though the horizontal momentum equation is similar to that found in hydrostatic σ-coordinate models, the system uses a perturbation pressure and not a height or geopotential, and of course a vertical momentum equation replaces the hydrostatic equation. In the new system the pressure equation takes the form

$$\frac{dp_1}{dt} = +\rho_o g w - \frac{c_p}{c_v} p \nabla \cdot \mathrm{V} + \frac{c_p p}{c_v T}(\frac{\dot{Q}}{c_p} + \frac{T_o}{\theta_o} D_\theta)$$

This pressure equation is formed in the same manner as the pressure equations (8), (9) and (10), and is not analogous to the surface pressure equation in a σ-coordinate hydrostatic model. The nonhydrostatic system is derived by direct substitution of the σ-coordinate, with $p^* = p^*(x,y)$, into the $\rho - T$ system which uses the pressure equation (10); the transformation is exact and the two systems are physically equivalent.

More in keeping with the use of σ-coordinates in hydrostatic models, Miller and White (1984) describe a σ-coordinate nonhydrostatic model in which p^* is the surface pressure. The fully compressible p and σ-coordinate equations are given by Miller and White. However, they are so unwieldy that they are not used without some simplifying approximations. The approximate set in p-coordinates used by Miller (1974), derived through a rigorous scale analysis, is shown by Miller and White to be the counterpart to the z-coordinate anelastic equations of Ogura and Phillips. The z and p-equation sets are not physically equivalent but the differences are at the order of approximation for both sets. The approximate σ-coordinate equation set is found to be a direct transformation of the pressure-coordinate nonhydrostatic set used by Miller.

In the σ and p coordinates of Miller and White, as in the z-coordinate anelastic equations, an elliptic equation must be solved for the geopotential which takes the place of pressure in the new systems. Integration of the p-coordinate system is simpler than integration of the σ system. In the p system, the momentum and thermodynamic equations are stepped forward to $t + \Delta t$ with some time-integration scheme and the geopotential at the new time is recovered by inverting an elliptic equation. This is the procedure used in most anelastic models. In the σ system, the solution procedure is much as it is for the hydrostatic σ system. However, instead of integrating the hydrostatic equation for the geopotential a 3-D elliptic equation must be solved. Also, the elliptic equation is significantly more complex than those arising in other anelastic systems.

Briefly summarizing, there are a variety of equation sets that can be employed in a nonhydrostatic model, along with a choice of several possible approximations to the fully compressible system. In terms of computational effort, most of the systems require a similar amount of time to integrate using the latest numerical techniques. Given this observation, it is not surprising that most current nonhydrostatic modelling efforts are focused on solving the fully compressible system.

3. METHODS FOR INTEGRATING THE FULLY COMPRESSIBLE SYSTEM

There are two methods for circumventing the severe timestep restriction brought about by the presence of acoustic modes in solutions to the fully compressible system. Either the terms responsible for the acoustic modes must by integrated implicitly (a semi-implicit model) or else explicitly with a smaller timestep within a time-split method (an explicit time-split model). In this section we will consider two models that are representative of the two classes, followed by a brief discussion of the stability of general semi-implicit and time-split methods.

Before continuing, we note that the presentation will focus on the temporal discretization of the systems. There exists much more leeway in the choice of spatial discretizations because they engender fewer stability restrictions. However, consideration of the variety of spatial discretizations existing in nonhydrostatic models is well beyond the scope of the present review. One general observation concerning the spatial discretization is that the majority of the nonhydrostatic models use a grid where the velocities u_i are staggered $1/2\Delta x_i$ in the i-th coordinate direction from the pressure and thermodynamic variables. This allows for accurate calculation of the divergence. We also note that only a few models are considered here. This is not meant to suggest that other models are more poorly constructed, but rather that the models highlighted were the first to incorporate the various schemes that are discussed here.

3.1 Semi-Implicit Methods

The model of Tapp and White (1976) was the first semi-implicit fully compressible nonhydrostatic model. The model numerics related to the dynamics have undergone significant modifications, in particular the introduction of terrain (Carpenter, 1979) and a modification of the semi-implicit scheme (Cullen, 1990). We will not consider terrain here other than to note that the choice of vertical coordinate systems does impact the structure of the Helmholtz equation and the effort needed for its solution. The following description of the model parallels that found in Cullen.

The model uses the perturbation variables Π_1 and θ_1 for the pressure and thermodynamic variable along with the geometric vertical-coordinate z. The momentum, thermodynamic and pressure equations can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + \theta_o \nabla \Pi_1 - \frac{g\theta_1}{\theta_o}\mathbf{k} = -\mathbf{V}\cdot\nabla\mathbf{V} - \theta_1 \nabla \Pi_1 + D,$$
$$\frac{\partial \theta_1}{\partial t} + w\frac{\partial \theta_o}{\partial z} = -\mathbf{V}\cdot\nabla\theta_1 + D_\theta + S_\theta,$$
$$\frac{\partial \Pi_1}{\partial t} - \frac{gw}{c_p\theta_o} + \frac{c_s^2}{c_p\theta_o}\nabla\cdot\mathbf{V} = R_\Pi,$$

where R_Π is given in Cullen. The acoustic-mode terms, along with the gravity-mode terms, are on the left side of the equations whereas the terms on the RHS of these equations are not responsible for the acoustic or gravity modes. The RHS terms are integrated explicitly using leapfrog time discretization. The terms on the lefthand sides of the equations are treated semi-implicitly. The temporal differencing for the equations is

$$\begin{aligned}
\mathbf{V}^{n+1} - \mathbf{V}^{n-1} &= -2\Delta t\theta_o(\alpha\nabla\Pi_1^{n+1} + (1-\alpha)\nabla\Pi_1^{n-1}) \\
&\quad + 2\Delta t\frac{g}{\theta_o}(\alpha\theta_1^{n+1} + (1-\alpha)\theta_1^{n-1})\mathbf{k} + 2\Delta t R_\mathbf{V}^n, \\
\theta_1^{n+1} - \theta_1^{n-1} &= -2\Delta t\frac{\partial\theta_o}{\partial z}(\alpha w^{n+1} + (1-\alpha)w^{n-1}) + 2\Delta t R_\theta^n, \\
\Pi_1^{n+1} - \Pi_1^{n-1} &= 2\Delta t\frac{g}{c_p\theta_o}(\alpha w^{n+1} + (1-\alpha)w^{n-1}) \\
&\quad - 2\Delta t\frac{c_s^2}{c_p\theta_o}(\alpha\nabla\cdot\mathbf{V}^{n+1} + (1-\alpha)\nabla\cdot\mathbf{V}^{n-1}) + 2\Delta t R_\Pi^n.
\end{aligned}$$

$R_\mathbf{V}$ and R_θ are the RHS terms in the original set and α is a temporal weighting factor for the semi-implicit step. Next, a second order correction term for Π_1,

$$\pi^n = 2(\alpha\Pi_1^{n+1} - \Pi_1^n + (1-\alpha)\Pi_1^{n-1}), \tag{17}$$

is used to eliminate Π_1^{n+1} from the system. The resulting equations are

$$\begin{aligned}
\mathbf{V}^{n+1} &= \mathbf{V}^{n-1} - 2\Delta t\theta_o(\nabla\pi^n) + 2\Delta t\frac{g}{\theta_o}(\alpha\theta_1^{n+1} + (1-\alpha)\theta_1^{n-1})\mathbf{k} \\
&\quad - 2\Delta t\theta_o(\nabla\Pi_1^n) + 2\Delta t R_\mathbf{V}^n,
\end{aligned} \tag{18}$$

$$\theta_1^{n+1} = \theta_1^{n-1} - 2\Delta t\frac{\partial\theta_o}{\partial z}(\alpha w^{n+1} + (1-\alpha)w^{n-1}) + 2\Delta t R_\theta^n, \tag{19}$$

$$\begin{aligned}
\pi^n &= 2\Delta t\frac{g}{c_p\theta_o}(\alpha w^{n+1} + (1-\alpha)w^{n-1}) \\
&\quad -4\Delta t\frac{c_s^2}{c_p\theta_o}(\alpha\nabla\cdot\mathbf{V}^{n+1} + (1-\alpha)\nabla\cdot\mathbf{V}^{n-1}) - 2(\Pi_1^n - \Pi_1^{n-1}) - 4\Delta t R_\pi^n
\end{aligned} \tag{20}$$

A Helmholtz equation for π is formed by eliminating θ^{n+1} from (18) using (19) and then eliminating $\mathbf{V}^{n+1}$ from (20) using (18), with the result being

$$\nabla_H^2 \pi^n + \left[\frac{\partial}{\partial z}\left(A^{-1}\frac{\partial}{\partial z}\right) - \frac{g}{c_s^2}A^{-1}\frac{\partial}{\partial z} - \frac{1}{4\alpha^2 c_s^2 \Delta t^2}\right]\pi^n = \Phi, \tag{21}$$

See Cullen (1990) for definitions of A and Φ. The advance a timestep one solves the Helmholtz equation (21) for π and then advances the momentum equations (18) and the thermodynamic equation (19).

The stability analysis of Cullen shows that the timestep is limited only by the advection speed; gravity waves do not limit the timestep and, of course, neither do the acoustic waves. The semi-implicit scheme without implicit treatment of the gravity waves is given in Tapp and White. They show that a stability criteria for that scheme is $N^2\Delta t^2 \leq 1$, where N is the Brunt-Väisälä frequency. The removal of this stability limitation is achieved by semi-implicit discretization of $g\theta_1/\theta_o\mathbf{k}$ in the vertical momentum equation and semi-implicit representation of the vertical advection of the base-state stratification $w\partial\theta_o/\partial z$ in the thermodynamic equation. Cullen notes that the base state should be chosen such that it is more stable than what is likely to be encountered in the simulation. This consideration is discussed in a hydrostatic context by Simmons *et al.* (1978)

The gravity-wave stability restriction in the Tapp and White model does not depend on the gridscale Δx. For typical values of N, the grid scale Δx would need to be at least a few tens of kilometers before the gravity-wave stability limit becomes more restrictive than that from advection (this assumes the use of an advection scheme with a general stability restriction of $U\Delta t \leq 1$). Cullen suggests that the removal of the stability restriction based on the stratification will allow use of semi-Lagrangian advection schemes and a much larger timestep.

3.2 Split-Explicit Methods

Klemp and Wilhelmson (1978) (hereafter referred to as KW) introduced a split-explicit technique for the integration of the fully-compressible system. The splitting technique is similar to, but predates, that used by Chao (1982) for hydrostatic models. A brief discussion of the stability of time-split techniques can be found in Section 3.3, and a detailed discussion of the stability of time-split numerical techniques for both hydrostatic and nonhydrostatic models is given in Skamarock and Klemp (1991). The semi-implicit treatment of gravity waves by Cullen has a natural time-split counterpart which we include in the overview of the KW model; it is described in detail in Skamarock and Klemp.

The momentum, thermodynamic and pressure equations used by KW are

$$\frac{\partial \mathbf{V}}{\partial t} + \theta_o \nabla \Pi_1 - \frac{g\theta_1}{\theta_o}\mathbf{k} = -\mathbf{V} \cdot \nabla \mathbf{V} - \theta_1 \nabla \Pi_1 + D, \tag{22}$$

$$\frac{\partial \theta_1}{\partial t} + w\frac{\partial \theta_o}{\partial z} = -\mathbf{V} \cdot \nabla \theta_1 + D_\theta + S_\theta, \tag{23}$$

$$\frac{\partial \Pi_1}{\partial t} + \frac{c_s^2}{c_p \theta_o^2}\nabla \cdot \theta_o \mathbf{V} = R_\pi. \tag{24}$$

This system is the same as that used by Tapp and White except here in the pressure equation the acoustically active term $(c_s^2/c_p\theta_o^2)\nabla \cdot \theta_o\mathbf{V}$ replaces $-gw/c_p\theta_o + (c_s^2/c_p\theta_o)\nabla \cdot \mathbf{V}$. As in Tapp and White, KW treat the terms on the RHS of (22) and (24) as acoustically inactive. These terms will be advance with a centered-in-time leapfrog step, this is referred to as the *large* timestep. The terms on the LHS of (22) and (24) are responsible for the acoustic modes and are advanced with a smaller, explicit timestep, this is the *small* timestep. As Cullen has demonstrated for the semi-implicit scheme, gravity-wave restrictions to the timestep can be circumvented by including the vertical advection of θ_o and the buoyancy term in the vertical momentum equation in the acoustic computations. Hence the appropriate terms in the thermodynamic equation (23) and the buoyancy term in the vertical momentum equation (22) are integrated along with the acoustically active terms.

The small and large timesteps are not taken separately in the KW scheme but rather are coupled. In keeping with the leapfrog approach, the acoustically active terms are integrated from $t - \Delta t$ to $t + \Delta t$ with n small timesteps $\Delta\tau$ where $\Delta\tau = 2\Delta t/n$. The temporal discretization for the system (22) through (24), with the small and large timesteps are coupled, is

$$u^{\tau+\Delta\tau} = u^\tau - \Delta\tau\frac{\partial \Pi^\tau}{\partial x} + \frac{2\Delta t}{n}R_u^t + \gamma_D\frac{\partial}{\partial x}(\nabla \cdot \theta_o\mathbf{V})^\tau \tag{25}$$

$$w^{\tau+\Delta\tau} = w^\tau - \Delta\tau\Big[(\alpha\frac{\partial \Pi_1^{\tau+\Delta\tau}}{\partial z} + (1-\alpha)\frac{\partial \Pi_1^\tau}{\partial z})$$

$$+ \frac{g}{\theta_o}(\alpha\theta_1^{\tau+\Delta\tau} + (1-\alpha)\theta_1^\tau)\Big] + \frac{2\Delta t}{n}R_w^t + \gamma_D\frac{\partial}{\partial z}(\nabla \cdot \theta_o\mathbf{V})^\tau \tag{26}$$

$$\theta_1^{\tau+\Delta\tau} = \theta_1^{\tau} - \Delta\tau(\alpha w^{\tau+\Delta\tau} + (1-\alpha)w^{\tau})\frac{\partial\theta_o}{\partial z} + \frac{2\Delta t}{n}R_\theta^t \tag{27}$$

$$\Pi_1^{\tau+\Delta\tau} = \Pi_1^{\tau} - \frac{\Delta\tau c_s^2}{c_p\theta_o^2}\nabla_H\cdot\theta_o\mathbf{V}^{\tau+\Delta\tau}$$
$$- \frac{\Delta\tau c_s^2}{c_p\theta_o^2}(\alpha\frac{\partial(\theta_o w^{\tau+\Delta\tau})}{\partial z} + (1-\alpha)\frac{\partial(\theta_o w^{\tau})}{\partial z}) + \frac{2\Delta t}{n}R_\Pi^t \tag{28}$$

Several features of this discretization deserve comment. First, the leapfrog integration of the non-acoustically active terms takes place through the addition of the terms $(2\Delta t/n)R_\phi^t$ in the discretized set. These RHS terms are computed before the small timesteps are taken. They are evaluated at time t and need only be evaluated once during the timestep; typically the RHS computations consume the bulk of overall CPU time. The horizontal momentum equation (25) and the pressure equation (28) are integrated using the forward-backward scheme of Mesinger (1977). In this application the horizontal momentum equations are advanced first using Π_1^τ followed by the pressure equation using the new velocities $\mathbf{V}_H^{\tau+\Delta\tau}$. For centered differencing on the C grid, the stability criteria for the horizontally propagating acoustic modes is $c_s\Delta\tau/\Delta x \leq 1$.

In typical simulations the vertical grid spacing is much finer than the horizontal grid spacing. Acoustic modes propagate at approximately the same speed vertically as horizontally, hence vertically propagating acoustic modes are handled implicitly in much the same manner as the acoustic modes are handled in the semi-implicit model of Tapp and White. The 1-D implicit system in the KW model is solved by eliminating $\Pi_1^{\tau+\Delta\tau}$ and $\theta_1^{\tau+\Delta\tau}$ from the vertical momentum equation (26) using (27) and (28). In the spatially discrete system the elimination leads to a tridiagonal matrix for $w^{\tau+\Delta\tau}$ which is easily solved. This removes all timestep restrictions arising from the vertically propagating acoustic modes and from the gravity waves.

Another feature of this explicit scheme is the introduction of an acoustic filter. The final terms in the temporally discrete momentum equations (25) and (26) involving derivatives of the divergence $\nabla\cdot\theta_o\mathbf{V}$ act to damp the acoustic modes in the system, hence we call this *divergence damping*. An analysis by Skamarock and Klemp shows that the addition of this term, involving the full 3-D divergence, effectively damps the acoustic modes and has little effect on the gravity waves. This filter is not the same as the the divergence damping found in some hydrostatic applications; the hydrostatic applications damp the horizontal divergence with the purpose of removing gravity waves (see Morel and Talagrand 1974, Sadourny 1975).

3.3 Stability Considerations

The preceeding fully-compressible models make use of what are now standard numerical methods for integrating the dynamical equations. The time discretization in both is based on the leapfrog scheme with the addition of an implicit scheme in the Tapp and White model and with the addition of a splitting technique which uses a forward-backward scheme of Mesinger in the split-explicit model of KW. An obvious question is whether there are any time-integration schemes that could replace leapfrog time differencing. A more accurate scheme may be desirable given that leapfrog with Asselin time-filtering is only first order accurate. Also, some of the forward-in-time advection schemes coupled with a split-explicit or semi-implicit approach would require only half the data storage of the original schemes, and in the case of the time-split approach would require only half the number of small timesteps to integrate the acoustic modes, needing now only to be advanced from t to $t + \Delta t$.

The feasibility of these extensions can be examined through a stability analysis of a simple acoustic-advective system.

$$u_t + c_s p_x + \mathrm{U} u_x = 0, \tag{29}$$

$$p_t + c_s u_x + \mathrm{U} p_x = 0. \tag{30}$$

Consider the application of the split-explicit integration technique of KW to the system (29) and (30) with Fourier spatial decomposition. The relevant discrete equations become

$$u^{\tau+\Delta\tau} = u^{\tau} - i\lambda_{cx} p^{\tau} - \frac{i}{\mathrm{n}_s}\lambda_u u^t \tag{31}$$

$$p^{\tau+\Delta\tau} = p^{\tau} - i\lambda_{cx} u^{\tau+\Delta\tau} - \frac{i}{\mathrm{n}_s}\lambda_u p^t, \tag{32}$$

where $\lambda_u = 2\Delta t k \mathrm{U}$, $\lambda_{cx} = \Delta\tau\, c_s k$ and k is the wavenumber. For the non-time-split system, i.e., $\Delta\tau = 2\Delta t$, the discrete system is pure leapfrog using forward-backward time differencing for the terms multiplied by c_s (Mesinger, 1977) and the method is stable for $|\lambda_u \pm \lambda_{cx}| \leq 2$ (see Figure 1a). In contrast, for acoustic modes alone the stability of the scheme requires $|\lambda_{cx}| \leq 2$ and for advection alone $|\lambda_u| \leq 2$. The stability of the scheme using several small timesteps per large timestep ($\Delta\tau = 2\Delta t/\mathrm{n}_s$, $\mathrm{n}_s > 1$), is shown in Figure 1b. It would appear that nothing is gained by taking several small timesteps per large timestep; large Courant numbers cannot be run for both the large and small (advective and acoustic) timesteps. The fingers of instability appearing in Figure 1b are related to the advection of acoustic modes

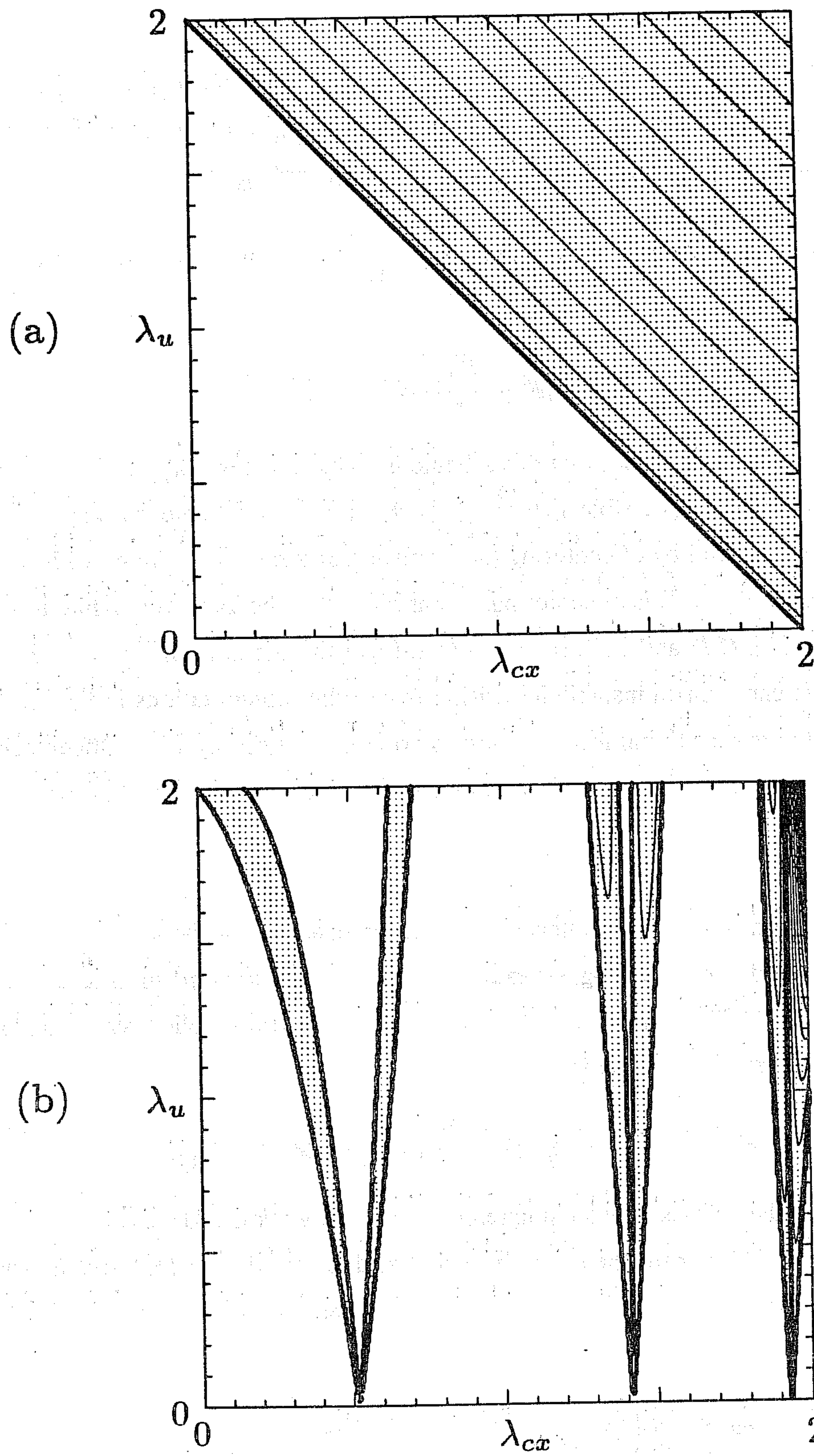

Figure 1. (a) Amplification factor from a von Neumann stability analysis for 1 small timestep per large timestep using the KW split-explicit scheme, (b) for six small timesteps per large timestep. Regions where $A > 1$ are stippled, the thick line is the 1.0 contour and the minor contours are at 0.2 intervals.

whose frequency is higher than that which can be represented on the large timestep. However, these unstable modes are well damped by the Asselin filter used in with the leapfrog scheme and are also controlled by the divergence damping used in the KW scheme.

Semi-implicit differencing of the system (29) and (30) leads to an uncoupled pair of discrete equations of the form

$$\phi^{n+1} = \phi^{n-1} - i\lambda_u\phi^n - \frac{i\lambda_{cx}}{2}\left(\phi^{n+1} + \phi^{n-1}\right).$$

Kwizak and Robert (1971) have shown that this scheme is stable whenever $\lambda_u^2 \leq 1 + \lambda_{cx}^2$. For acoustic modes in the fully compressible system $\lambda_{cx} \gg \lambda_u$, hence the scheme is always stable. Further stability is engendered by offcentering the implicit portion of the scheme with more weight given to the value ϕ^{n+1}. This offcentering, accomplished in the Tapp and White model by using $\alpha > 1/2$ in (16), (17) and (18), serves to transform the neutral scheme to one that damps. This damping can control instabilities arising from other computations in the model and is also used in the vertically implicit small-timestep computations by KW. Offcentering the implicit computations does degrade the accuracy of the overall scheme to first order in time.

Returning to the question of the use of alternate time discretizations for the advection terms that are presently handled using leapfrog, we examine the use of the forward-in-time Crowley schemes (Tremback *et al* 1987) coupled with the split-explicit and semi-implicit schemes. The second-order accurate Crowley scheme is

$$\phi_j^{t+\Delta t} = \phi_j^t - \frac{\beta}{2}\left(\phi_{j+1}^t - \phi_{j-1}^t\right) + \frac{\beta^2}{2}\left(\phi_{j+1}^t - 2\phi_j^t + \phi_{j-1}^t\right), \quad (33)$$

where β is $U\Delta t/\Delta x$. In the split-explicit technique, the discrete equations (31) and (32) for the simple system (29) and (30) remain the same. The spatial discretization in (33) and Fourier decomposition in x lead to a new definition for the parameters λ_u and λ_{cx} in the discrete equations,

$$\lambda_u = -i\beta^2(1 - \cos(k\Delta x)) + \beta\sin(k\Delta x). \quad (34a)$$

$$\lambda_{cx} = \frac{2c_s\Delta\tau}{\Delta x}\sin\frac{k\Delta x}{2} \quad (34b)$$

Figure 2 depicts the stability region for 6 small timesteps per large timestep with explicit integration of the acoustic modes for the $4\Delta x$ horizontal wavelength mode. In the absence

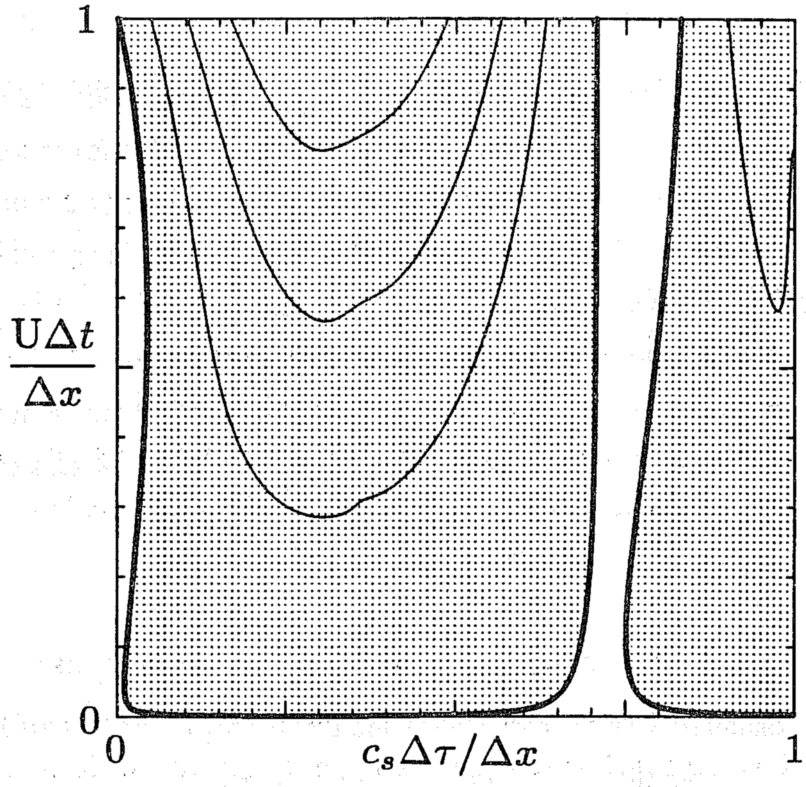

Figure 2. Amplification factor for the $4\Delta x$ mode from a von Neumann stability analysis for 6 small timesteps per large timestep using the second order Crowley scheme for advection in the split-explicit KW method on a C-grid. Contoured as in Figure 1.

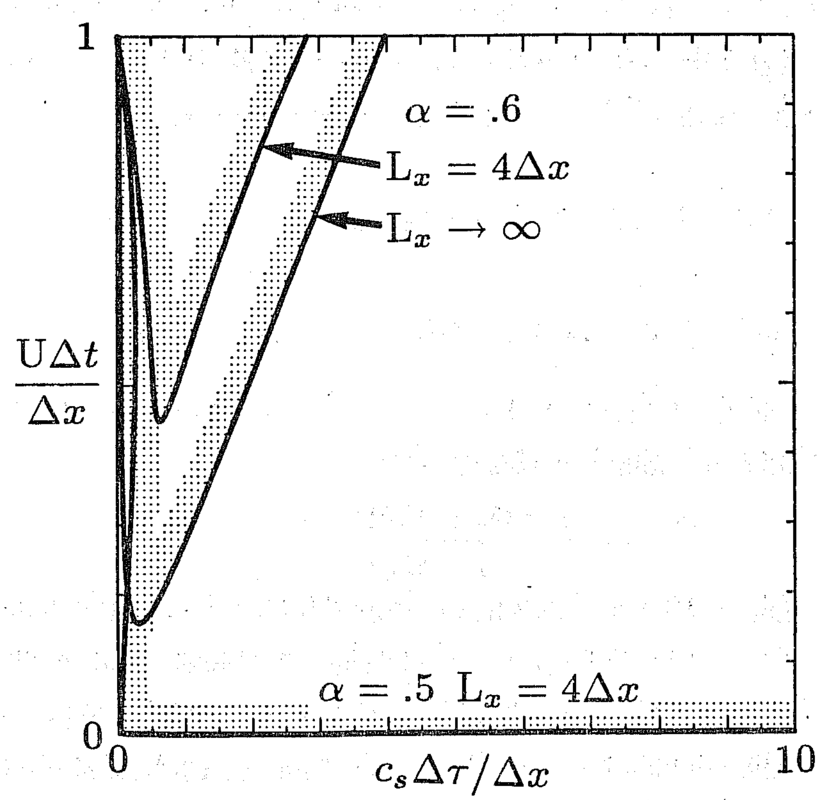

Figure 3. Amplification factor from a von Neumann stability analysis of a semi-implicit scheme using second order Crowley scheme for advection. The finite differencing is on a C-grid. Contoured as in Figure 1.

of acoustic modes the Crowley scheme is stable for $\beta \leq 1$. The instabilities associated with the advection of high frequency acoustic modes are significantly stronger for this scheme than for the leapfrog scheme (compare Figs. 2 and 1b). Divergence damping and other filters will help control the instabilities, particularly those associated with the short horizontal wavelength modes. However, the longer wavelength modes are also unstable, and while the instability is weaker for longer wavelengths, it covers a larger portion of the (λ_{cx},λ_u)–stability domain. The longer wavelength modes are not appreciable damped by most filters, hence the use of this Crowley scheme with explicit integration of the acoustic modes is not feasible. We have also performed numerical experiments with the advection scheme of Smolarkiewicz (1984) and find similar results when used in a split-explicit model.

The stability of the higher-order Crowley schemes have been examined for use in the explicit time-split algorithm. Skamarock and Klemp (1991) find that the higher-order Crowley schemes excite weaker instabilities than their lower-order brethren. In general, the increased damping in the lower-order schemes on the large timestep leads to larger instabilities associated with the advection of the acoustic modes. The higher order Crowley schemes have less damping, and smaller instabilities, and the counter-intuitive result is that the scheme with less damping is more stable. Through sixth order, however, the instabilities are still more severe than those arising in the KW scheme, and the long wavelength instabilities remain.

Implicit integration of the acoustic modes can be examined by considering the simple discretized equation

$$\phi^{n+1} = \phi^n - i\lambda_u\phi^n - \frac{i\lambda_{cx}}{2}(\alpha\phi^{n+1} + (1-\alpha)\phi^n). \tag{35}$$

where λ_u and λ_{cx} are given in (34a,b). For the second order Crowley scheme, the amplification factor in the von Neumann stability analysis for (35) is

$$A = \frac{1 - i\lambda_w - i\lambda_{cz}(1-\alpha)}{1 + i\lambda_{cz}\alpha}.$$

Figure 3 shows the stability space for this scheme. Both short and long wavelength instabilities exist, though both are effectively damped by offcentering the implicit acoustic mode integrations. The higher order Crowley schemes possess these same stability characteristics when coupled with implicit integration of the acoustic modes. Thus the Crowley advection schemes can be used in models where the acoustic modes are integrated implicitly, but the acoustic integration must be offcentered for stability.

This result for the 2nd-order Crowley scheme also extends to the 2nd-order semi-Lagrangian scheme of Bates and Macdonald (1982).

3.4 Discussion

In this section we have outlined the semi-implicit scheme of Tapp and White and the split-explicit scheme of Klemp and Wilhelmson. Furthermore, we have considered possible extensions to the two approaches by means of replacing the leapfrog time differencing with some more sophisticated forward-in-time integration schemes. We search for more accurate and possibly less costly (memory and/or CPU) schemes. Unfortunately, the stability of the forward-in-time integration schemes is poor when coupled with semi-implicit or split-explicit integration of the acoustic modes.

The stability results presented in Section 3.3 are further reinforced by attempted applications of the more sophisticated integration schemes in fully compressible models. Models in which the momentum equations are integrated with schemes other than leapfrog have, on the whole, proven unstable. Presently, most models use *hybrid* schemes wherein the momentum equations are integrated with the traditional leapfrog-based semi-implicit or split-explicit schemes and the scalars are integrated using some more accurate forward-in-time scheme. The hybrid models are stable.

Within the leapfrog framework many types of advection schemes can be used. For example, Tanguay *et al* (1990) present a semi-implicit, semi-Lagrangian fully compressible model. This model handles the gravity-wave terms implicitly and can be considered the semi-Lagrangian counterpart to the semi-implicit model of Tapp and White. The model is developed in a leapfrog time-differencing framework and appears quite stable, possibly a result of its use of the leapfrog scheme. As reported in Tanguay *et al*, a timestep of 1800s is used in a test with a large horizontal gridscale (127 km). This is about twice the timestep as would typically be used in the semi-implicit or split-explicit nonhydrostatic models described previously and the tests illustrate the advantage of the semi-Lagrangian scheme, i.e., the ability to use larger timesteps.

The model of Tanguay *et al* is semi-Lagrangian in only the horizontal directions; vertical advection is centered in space and time. At this point it is worth considering in more detail how a nonhydrostatic model is to be used. As suggested by scaling arguments and revealed through practical experience, hydrostatic models are adequate when used on gridscales no smaller than 10 to 20 km. Nonhydrostatic models should be used when gridscales are 10 to 20 km or less. For example, the British Meteorological Office uses a nonhydrostatic model on a 15 km grid. Simulations of strong convection with cloud models using a gridscale of 2 km often show updrafts of 30 to 40 ms^{-1}. Also, when the horizontal gridscale is a few kilometers the vertical velocities become equal in magnitude to the horizontal velocities and the circulations in the vicinity of the convection have an aspect ratio H/L of order 1. From a practical standpoint, it is the vertical advection that limits the maximum timestep in the cloud models. For modelling philosophy, these observations strongly suggest the necessity of designing the horizontal and vertical discretization as a whole and not as separate pieces as is the general practice in hydrostatic modelling. In the semi-Lagrangian model of Tanguay *et al* one can expect that the the larger timesteps allowed by the scheme may not be realized when used with a gridscale of 10 km or less because the vertical velocity will limit the timestep. It is not clear how straightforward the extension of the semi-Lagrangian method to three dimension would be. Smolarkeiwicz and Pudykeiwicz (1991) have constructed a semi-Lagrangian 2-D (x, z), incompressible Boussinesq model and have extended it to three dimensions (Smolarkeiwicz, personal communication). The method has not been extended to the fully compressible system.

Finally, it is worth noting that the acoustic modes are treated poorly in these schemes. The semi-implicit schemes propagate the modes more slowly, and when the implicit scheme is offcentered the all modes, including the acoustic modes, are damped. The split-explicit scheme advances the modes more accurately in the horizontal, but is vertically implicit and may damp modes with vertical structure. In addition, the divergence damping technique used in the latest version of the KW model serves as an acoustic filter and drives the solution to a nondivergent, but not anelastic, state. While the analysis of Skamarock and Klemp suggest that the filtering has little effect on the gravity modes, we are continuing to investigate the impact of the various acoustic filters.

4. SOLUTION TECHNIQUES FOR ELLIPTIC EQUATIONS

The discretized anelastic equations and the semi-implicit methods for the fully compressible equations lead to elliptic equations for some form of pressure. While the equations are linear, they are elliptic in three dimensions and the coefficient structure is often non-uniform, particularly when orographic effects are included in the model. The solution techniques for the systems can be broken up into two types; direct methods and iterative methods. In the case of non-constant coefficients iterative approaches can be coupled with the direct solution methods.

The anelastic model of Clark (1977) incorporates a solution method for an elliptic pressure equation that features many of the common manipulations used in other models. The diagnostic pressure equation used in the model can be expressed as

$$\nabla_H^2 G^{1/2}p + \frac{1}{G^{1/2}}\frac{\partial^2}{\partial z^2}P = -\frac{\partial^2(G^{1/2}G^{13}p)}{\partial x\partial z} - \frac{\partial^2(G^{1/2}G^{12}p)}{\partial y\partial z}$$
$$-\frac{1}{G^{1/2}}\frac{\partial}{\partial z}\left[G^{1/2}G^{13}\left(\frac{\partial G^{1/2}p}{\partial x} + \frac{\partial G^{1/2}G^{13}p}{\partial z}\right) + G^{1/2}G^{23}\left(\frac{\partial G^{1/2}p}{\partial y} + \frac{\partial G^{1/2}G^{23}p}{\partial z}\right)\right]$$
$$+ F \tag{36}$$

where $G^{1/2}$ is the Jacobian of the transformation and G^{ij} is the conjugate tensor (see Gal-Chen and Sommerville, 1975). The approach taken by Clark involves solving (36) iteratively holding the RHS terms fixed during each iteration. For this iterative scheme (36) is rewritten as

$$G\nabla_H^2 P^{n+1} + \frac{\partial^2}{\partial z^2}P^{n+1} = LP^n + F, \tag{37}$$

where P^n is the n^{th} iterative solution for $G^{1/2}p$. The terms on the RHS of (36) are evaluated using the latest iterative solution of P, thus only a simple elliptic equation need be solved. In practice only 2 to 3 iterations using (37) are required for convergence.

Next, to solve the simpler 3-D elliptic equation (37) during any iteration the dimension of the system is reduced using the method of Ogura (1969). Vertical discretization of (37) results in

$$G\nabla_H^2\phi + A\phi = H, \tag{38}$$

where A is an $(m \times m)$ matrix for an m level model and ϕ is the column vector $\phi^{\text{T}} = [\phi_1, \phi_2, ...\phi_m]$. (38) represents a coupled set of 2-D elliptic equations. Diagonalization of the matrix A is given by $\Lambda = SAS^{-1}$, where Λ and S are the eigenvalue and eigenvector matrices for A. The diagonalization can be used to derive an uncoupled set of 2-D elliptic equations from (38) with the result being

$$G\nabla_H^2\Phi + \Lambda\Phi = R, \tag{39}$$

where $\Phi = S\phi$ and $R = SH$. The matrix A is constant and does not depend on x or y, hence the eigenvalues and eigenvectors need only be computed and stored at the beginning of a computation.

Finally, we are left with the need to solve 2-D elliptic equations with simple coefficient structure. Both direct and iterative methods may be used. Clark uses a direct Fourier method. Tapp and White use the iterative Alternating Direction Implicit (ADI) method for solving their counterpart to (39). Tanguay *et al* use Successive Over Relaxation (SOR) on the fully 3-D elliptic equation. The Tapp and White model with terrain (Carpenter, 1979) does not require the outer iteration required in Clark's model [see (37)] and neither does the model of Tanguay *et al.*

There are two points to consider when choosing the form of the the diagnostic equations and their solution techniques. In order to decompose a 3-D elliptic equation into a system of uncoupled 2-D equations it is critical that the vertical discretization of the LHS term in the system not possess coefficients that are dependent on the horizontal position. One does not want to carry a different matrix A for every horizontal gridpoint. Clark accomplishes this by using an outer iteration for the terms that would be troublesome (Eq. 37) while Carpenter accomplishes this through the choice of the vertical coordinate, using $\eta = z - z_s$ as opposed to the transformed z coordinate given in Section 2.3 and used by Clark. The horizontal coefficient structure matters little if iterative methods for solving the 2-D elliptic systems are used, though the use of direct methods require that the coefficients be regular.

We conclude by noting that the solution techniques used for the elliptic systems have existed for quite some time, and more recent, possibly more efficient, techniques have yet to be exploited. In particular, multigrid methods (Fulton *et al*, 1986) have not been applied in the semi-implicit fully-compressible models where they would replace existing iterative techniques. Multigrid methods have been used in the Clark model but have not, at present, proven as efficient as the iterative/direct method outlined here (Clark, personal communication).

5. SUMMARY

We have presented an brief review of methods used to solve the nonhydrostatic equations of motion. Acoustic modes present the most formidable numerical problems in designing non-hydrostatic models. Anelastic approximations, discussed in Section 2.2, remove the acoustic modes from the system and leave a straightforward set of prognostic equations along with a complex 3-D elliptic equation for diagnosing the pressure. Solution techniques for solving this diagnostic equation, considered in Section 4, rely on iterative techniques, possibly in combination with direct methods, along with order reduction.

Anelastic models were the only nonhydrostatic models in use from the early 1960's through the mid 1970's. Two approaches for integrating the fully compressible equations that overcome the difficulties associated with the acoustic modes were introduced in the mid-1970's. The first integrates the non-acoustic terms explicitly while treating the acoustic terms implicitly. This semi-implicit approach is considered in Section 3.1. The second method integrates the acoustic terms with a smaller timestep than the other terms. This is the split-explicit approach and is discussed in Section 3.2. In both approaches the terms responsible for gravity waves can be included in the acoustic integrations at little extra cost, hence the resulting schemes possess timesteps limited only by the advection. The primary difference between the two approaches is that the semi-implicit techniques leads to a diagnostic 3-D elliptic equation for a pressure correction while the split-explicit approach requires taking multiple, explicit small timesteps for the acoustic mode integration.

The fully compressible models have been undergoing a slow evolution since they were first developed in the mid to late 1970's, having incorporated only modest refinements since the mid 1970s. Possible extensions to the present time-split or semi-implicit schemes possess stability problems which have yet to be overcome. In particular, the use of more sophisticated advection schemes has been limited to scalar advection in the hybrid models.

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