LATERAL AND UPPER BOUNDARY CONDITIONS

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1. INTRODUCTION

Lateral and upper boundary conditions are a major consideration in limited area modelling. In his pioneering work Charney et al. (1950), Charney (1962) specified $\vec{V}_n$ everywhere on the boundary and other variables only at inflow points. Specification here means that data is derived from a source external to the information within the integration domain. Davies (1973) applied the energy method (see also Oliger and and Sundström, 1978) to obtain sufficient conditions for the uniqueness of the solutions on a limited area. For the idealized systems considered he shows that to specify $\vec{V}_n$ everywhere on the boundary and $\vec{V}_t$, θ, etc only at inflow points leads to unique solutions. However, we should keep in mind that in this type of analysis the conditions suggested are only a sufficient mathematical condition which limits the boundedness of the solution. The condition is not unique nor is it necessarily in general physically correct for the problem at hand.

Simple hyperbolic systems of equations can be analyzed in terms of their Riemann invarients and characteristic velocities. The Riemann invarients are the new field variables resulting from a diagonalization of the linear system of equations. Sundström and Elvius (1979) argue that to insure a well posed solution one should specify only those invarients whose characteristic velocity is directed into the domain. All others should be treated using some sort of extrapolation from the interior. Thus, the number of fields to be specified is equal to the number of inwardly directed characteristic velocity components. Any larger a set of specification is considered over specification. Thus, specification of $\vec{V}_n$ at all boundaries is in general over specification since there will be characteristics associated with the normal velocities directed outward from the domain somewhere on the boundary.

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A Riemann invarient analysis is not a tractable under taking for models of any complexity. As a result modellers have strayed rather far from the formal boundary condition specifications prescribed by formal matrix analysis. Probably all limited area meteorological models overspecify their boundary conditions to some extent. Various boundary zone remedies are employed as a result. Perhaps the pseudo–radiation condition derives its current popularity from the fact that $\vec{V}_n$ specification can be more or less limited to inflow boundaries and this has a certain physical attractiveness. The reviews of Miller and Thorpe (1981) and Hedley and Yau (1988) suggest that this type of boundary condition has perhaps been the most successful to date in small scale limited area modeling.

All models, even global, have to consider the upper boundary as $z \to \infty$. Planetary waves on down to gravity waves can transmit energy vertically and without a proper treatment of this boundary physically erroneous solutions will be obtained.

This paper briefly reviews some of the physical and mathematical concepts used to derive boundary conditions and then describes some of the recent treatments of upper and lateral boundary conditions.

2. BASIC CONCEPTS

Linear equations of motion in three spatial dimensions will be used to present some of the basic concepts used in the formulation of boundary conditions for limited area models. The anelastic system of equations in the Cartesian framework will be adopted although this should not affect any of the present conclusions. The equations of motion are taken as

$$\frac{Du}{Dt} + wU_z = -\frac{\partial}{\partial x}(p/\rho) \tag{1}$$

$$\frac{Dv}{Dt} + wV_z = -\frac{\partial}{\partial y}(p/\rho) \tag{2}$$

$$\mu\frac{Dw}{Dt} = -\frac{\partial}{\partial z}(p/\rho) + g\theta \tag{3}$$

and the first law of thermodynamics as

$$\frac{D\theta}{Dt} = -Sw. \tag{4}$$

In (3) μ is a mathematical tracer. When μ =1 the system is non–hydrostatic and when μ =0 it is hydrostatic. In the anelastic system, mass continuity takes the form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0. \tag{5}$$

In the above we define

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} \tag{6}$$

and $\rho = \rho(z)$, $U = U(z)$ and $V = V(z)$ and $S = \frac{d\ell n\bar{\theta}}{dz} = S(z)$.

The Bruint-Väisälä frequency N is given as $N^2 = gS$. In this system the total value of the various field variables is defined as

$$\begin{aligned} u_{tot} &= u + U \\ v_{tot} &= v + V \\ w_{tot} &= w \\ \theta_{tot} &= \bar{\theta}(1 + \theta) \end{aligned} \tag{7}$$

Equations (1) through (5) can be combined to form the wave equation

$$\left(\frac{D}{Dt}\right)^2 [\mu\nabla_H^2 w + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\rho w\right)] + \frac{D}{Dt}(\sigma w_x U_z + \sigma w_y - V_z w_x U_{zz} - w_y V_{zz}) + N^2\nabla_H^2 w = 0. \tag{8}$$

For the present purpose it is simplest to assume ρ is constant (Boussinesq approximation) which results in $\sigma = \frac{d\ell n\rho}{dz} = 0$ and also to consider cases without wind shear. Under these assumptions (8) reduces to

$$\left(\frac{D}{Dt}\right)^2 \left(\mu\nabla_H^2 w + w_{zz}\right) + N^2\nabla_H^2 w = 0 \tag{9}$$

which is sufficient to solve for w given initial and boundary conditions.

a) Dispersion Relation

The dispersion relation expresses the frequency as a function of the wavenumber components. From this relationship the phase and group velocities can be derived.

Let

$$w = \hat{w}e^{i(kx+\ell y+mz-\omega t)} \tag{10}$$

and substituting (10) into (9) results in the dispersion relation for the present system of equations

$$\omega = kU + \ell V \pm \sqrt{\frac{\kappa_H^2 N^2}{\kappa^2}}, \tag{11}$$

where $\kappa_H^2 = k^2 + \ell^2$ and $\kappa^2 = \mu\kappa_H^2 + m^2$. The fact that (11) has two solutions at each point in wavenumber space can be physically interpreted to mean there are waves with identical wavenumbers travelling in opposite directions.

b) Phase and Group Velocities

The phase velocity is the velocity at which the phase of the wave moves whereas the group velocity is the velocity at which the energy or wave packet moves. These quantities are given as

$$\vec{c}_{ph} = (\omega/k, \omega/\ell, \omega/m) \tag{12}$$

$$\vec{c}_g = (\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial \ell}, \frac{\partial\omega}{\partial m}). \tag{13}$$

In terms of lateral and upper boundary conditions it is the direction of the group velocity which is of prime importance because this describes the direction in which energy or information is travelling. Consider the two examples of the lateral boundary and the upper boundary.

At $x = 0$, the group and phase velocities in the lateral direction are

$$c_g = \frac{\partial\omega}{\partial k} = U \pm \frac{Nkm^2}{\kappa^2\sqrt{\kappa^2\kappa_H^2}} \tag{14}$$

$$c_{ph} = \frac{\omega}{k} = U \pm \frac{N}{k}\sqrt{\frac{\kappa_H^2}{\kappa^2}} \tag{15}$$

and since components traveling out of the domain are desired the bottom or "-" sign is chosen. In the case of two–dimensional hydrostatic motion we find $c_g = c_{ph} = U - N/m$. thus, when $N/m > U$ these wave components will be travelling upstream and require treatment at the $x = 0$ or advection inflow boundary.

Now in the vertical, the group and phase velocities are

$$c_g = \frac{\partial\omega}{\partial m} = \mp\frac{Nm}{\kappa^2}\sqrt{\frac{\kappa_H^2}{\kappa^2}} \tag{16}$$

$$c_{ph} = \frac{\omega}{m} = \frac{(kU + \ell V)}{m} \pm \frac{N}{m}\sqrt{\frac{\kappa_H^2}{\kappa^2}} \tag{17}$$

and again the bottom or "+" sign is chosen to give energy travelling out of the domain. In the case of two–dimensional hydrostatic motion we find

$$c_g = \frac{kN}{m^2} \tag{18}$$

$$c_{ph} = \frac{kU}{m} - \frac{kN}{m^2} \tag{19}$$

and this time the group velocity is in the opposite direction to that of the intrinsic phase velocity. This fundamental difference between the phase and group velocity directions at the upper and lateral boundaries has led to a somewhat distinct treatment between the two problems.

c) Pseudo-differential operators

Enquist and Majda (1977, 1979) presented a technique to the fluid dynamics community which can be used to derive boundary conditions which are local both in time and space. This is an extremely important practical consideration in the design of boundary conditions. The present procedure has most recently been applied by Rasch (1986) to the treatment of gravity and Rossby waves in GCM modelling. His work will be discussed in more detail later.

Consider (9) in its two–dimensional and hydrostatic form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \tilde{w}_{zz} - k^2N^2\tilde{w} = 0 \tag{20}$$

where

$$w = \sum_k \tilde{w}(k, z, t)e^{ikx}. \tag{21}$$

Now (20) can be written as

$$\tilde{w}_{zz} - k^2\frac{N^2}{D^2}\tilde{w} = 0 \tag{22}$$

where $D = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$. Equation (22) can be factorized as

$$\left(\frac{\partial}{\partial z} + \frac{kN}{D}\right)\left(\frac{\partial}{\partial z} - \frac{kN}{D}\right)\tilde{w} = 0 \tag{23}$$

giving the two solutions

$$\tilde{w}_z \pm \frac{kN}{D}\tilde{w} = 0. \tag{24}$$

Substitution of (24) into the horizontal derivative of (1) results in

$$\tilde{p} = \frac{\rho N \tilde{w}}{k} \tag{25}$$

where the plus sign is chosen to select outward propagating waves. Equation (25) is the boundary condition proposed by Bougeault (1983) and Klemp and Durran (1983). This boundary condition will be discussed again later.

A more general application of pseudo–differential operators is discussed in Enquist and Majda (1977) with respect to the two–dimensional equation

$$\frac{\partial^2}{\partial t^2}w - \frac{\partial^2}{\partial x^2}w - \frac{\partial^2}{\partial y^2}w = 0 \tag{26}$$

and the object is to find a boundary condition at an x=constant boundary. (26) is rewritten as

$$\Big(\frac{\partial}{\partial x} - \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\Big)\Big(\frac{\partial}{\partial x} + \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\Big)w = 0 \tag{27}$$

resulting in the two solutions

$$\Big(\frac{\partial}{\partial x} \pm \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\Big)w\Big|_{x=0} = 0$$

or

$$\Big(\frac{\partial}{\partial x} \pm \frac{\partial}{\partial t}\sqrt{1 - \frac{\frac{\partial^2}{\partial y^2}}{\frac{\partial^2}{\partial t^2}}}\Big)w\Big|_{x=0} = 0 \tag{28}$$

as the boundary condition. However, (28) is still a touch abstract for application to the average model code.

The following well-posed approximations of $\sqrt{1-X}$ (where $X = \frac{\partial^2}{\partial y^2}/\frac{\partial^2}{\partial t^2}$) to zero and first order Taylor series approximation and second order Páde approximation are

$$\begin{aligned} \sqrt{1-X} &= 1 \\ \sqrt{1-X} &= 1 - \frac{X}{2} \\ \sqrt{1-X} &= \frac{2 - \frac{X}{2}}{2 + \frac{X}{2}} \end{aligned} \tag{29}$$

$$c_{ph} = \frac{\omega}{m} = \frac{(kU + \ell V)}{m} \pm \frac{N}{m}\sqrt{\frac{\kappa_H^2}{\kappa^2}} \tag{17}$$

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which lead to the respective boundary conditions,

$$
\begin{aligned}
\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\Big)w\Big|_{x=0} &=0\\
\Big(\frac{\partial^2}{\partial x\partial t}-\frac{\partial^2}{\partial t^2}+\frac{1}{2}\frac{\partial^2}{\partial y^2}\Big)w\Big|_{x=0} &=0\\
\Big(\frac{\partial^3}{\partial x\partial t^2}-\frac{1}{4}\frac{\partial^3}{\partial x\partial y^2}-\frac{\partial^3}{\partial t^3}+\frac{3}{4}\frac{\partial^3}{\partial t\partial y^2}\Big)w\Big|_{x=0} &=0
\end{aligned}
\tag{30}
$$

with the following respective wave reflection coefficients

$$
\begin{aligned}
r &= \left|\frac{cos(\phi)-1}{cos(\phi)+1}\right|\\
r &= \left|\frac{cos(\phi)-1}{cos(\phi)+1}\right|^2\\
r &= \left|\frac{cos(\phi)-1}{cos(\phi)+1}\right|^3
\end{aligned}
\tag{31}
$$

where ϕ is the angle of incidence of a single wave component. The authors also describe how one can apply this theory to non-homogeneous linear wave equations using similar factoring techniques.

3. UPPER BOUNDARY CONDITIONS

Rasch (1986) lists a number of UBCs suggested by modellers. Four of these are

[B1] Setting w (in either p or z coordinates) to zero at some finite height

[B2] Setting w to zero at p=0

[B3] Choosing a boundary layer which absorbs all energy entering it.

[B4] Choosing a condition which allows energy to radiate outward at some finite height

Boundary condition [B1] which is effectively a solid wall condition results in total energy reflection and [B2] behaves similarly (Lindzen et al., 1968). The reason [B2] behaves like [B1] is due to the exponential mapping (Grosch and Orszag, 1977) from p to z coordinates. Waves of finite vertical wavelength will eventually reach a height where they are under resolved and reflected. Rasch (1986) points out that

[B2] is used in virtually all complex large–scale numerical models of the atmosphere today. A number of idealized studies such as Bates (1977), Shutts (1978) and Chen and Trenberth (1985) have found the tropospheric solution to be very sensitive to the formulation of the UBC.

[B3] is the 'sponge layer' formulation. In order for the sponge layer to be effective it must resolve the wave reasonably well (8 to 10 grid points per wave length) and extend over at least one wavelength. As a result, absorbers of this type are considered to be very expensive. However, this is not necessarily the case for models using two–way interactive nesting. The absorber region can employ about half the spatial resolution in the three spatial directions compared to that used over the main area of interest. Combine this with twice longer time steps for the coarse mesh then the overhead for such sponge type absorbers is closer to the 10 % level. This is the approach used in the model of Clark and Hall (1991).

Sponge layers can be of either the Rayleigh friction and Newtonian cooling type or of the diffusive type where either a horizontal and/or vertical eddy mixing coefficient is employed. There is a certain level of difficulty in tuning the diffusive absorber because of the broad range of horizontal and vertical wavelengths in the general three–dimensional problem. The range of group velocities further complicates the design of the diffusive absorber. Klemp and Lilly (1978) use this type of absorber in their two–dimensional mountain wave study and present an analysis on the design of the absorber using horizontal mixing. Peltier and Clark (1979) also used such an absorber but with both horizontal and vertical mixing active.

Rayleigh friction and Newtonian cooling is typically treated using a height dependent τ where the field variable ϕ is treated as

$$\frac{\partial \phi}{\partial t} + ... = -\frac{1}{\tau}\phi'. \tag{32}$$

$\frac{1}{\tau}$ typically varies from zero at the bottom of the absorber layer and monotonically increases to its maximum at the model top. The maximum amplitude of $\frac{1}{\tau}$ is typically chosen so that the dominant waves are damped to, say, e^{-2} of their initial value upon entering the absorbing layer. In this case, waves of all scales decay at the same rate.

However, the broad range in group velocities for fully three-dimensional flow affects the time available for decay and makes the Rayleigh friction absorber also difficult to tune. Jones and Houghton (1971), Clark (1977), Durran and Klemp (1983), Hoinka (1985), Clark and Farley (1984), among others, used the Rayleigh friction type absorber in their gravity wave studies.

Boundary condition [B4] has been used in a number of studies to allow both gravity and Rossby waves to exit the model top. In describing the use of pseudo-differential operators, the boundary condition of Bougeault (1983) and Klemp and Durran (1983)

$$\tilde{p} = \frac{\rho N \tilde{w}}{|\kappa_H|} \tag{33}$$

was derived where $\tilde{p}$ and $\tilde{w}$ are the horizontal Fourier components of p and w. The major assumptions under lying (33) is that the flow is hydrostatic, Boussinesq with no Coriolis force. It has been found to be quite useful in the application to small scale gravity wave flow. However, for larger scale problems one needs to consider both gravity waves as well as waves involving Coriolis effects. One also needs to eventually consider the spherical geometry of the earth. As a first good step in this direction Rasch (1986) developed an UBC for a linear β–plane PE model which considers both gravity and Rossby waves. He also demonstrates the utility of the scheme in an application to nonlinear quasi–geostrophic model.

The linear equations of Rasch's model are

$$(i\sigma - i\epsilon/2)\tilde{\chi} - \tilde{\psi} = -\tilde{\phi}/f \tag{34}$$

$$(i\sigma - i\epsilon/2)\tilde{\psi} + \tilde{\chi} = 0 \tag{35}$$

$$i\sigma\tilde{\phi}/f - ghF\tilde{\chi} = 0 \tag{36}$$

where $\tilde{\chi}$, $\tilde{\psi}$ and $\tilde{\phi}$ are the Fourier coefficients of the velocity potential, streamfunction and geopotential, respectively. $\epsilon = f_y k/(k^2 + \ell^2)$ where k is the zonal and ℓ the meridional wavenumber. $F = m^2/f^2$ where $m^2 = k^2 + \ell^2$ and h is a separation constant given by

$$gh = \frac{\sigma f^2}{m^2} \frac{[(\sigma - \epsilon/2)^2 - 1]}{(\sigma - \epsilon/2)} \tag{37}$$

which is used to distinguish between gravity and Rossby waves. Now $\tilde{\chi}$, $\tilde{\psi}$, $\tilde{\phi}$ as well as $\tilde{w}$ are functions of z and t such that

$$\tilde{\chi} = \hat{\chi} V(z) e^{i\sigma f t} \tag{38}$$

where

$$V_{zz} - V_z + \frac{S}{gh} V = 0 \tag{39}$$

which has a solution of the form

$$V = V_+ e^{(\frac{1}{2}+in)z} + V_- e^{(\frac{1}{2}-in)z} \quad \text{for} \quad \mathrm{n}^2 > 0 \tag{40}$$

and

$$V = U_+ e^{(\frac{1}{2}+\mu)z} + U_- e^{(\frac{1}{2}-\mu)z} \quad \text{for} \quad \mathrm{n}^2 < 0 \tag{41}$$

where n is the index of refraction and is given by

$$n = sgn(\sigma)(S/gh - 1/4)^{\frac{1}{2}} \tag{42}$$

and

$$\mu = (1/4 - S/gh)^{\frac{1}{2}}. \tag{43}$$

The V_+ term in (40) corresponds to upward propagating energy and the analytical boundary condition is given as

$$\xi_z = (1/2 + in)\xi \tag{44}$$

where ξ is any of (χ, ψ, ϕ, w). The Rossby regime corresponds to $\sigma \approx \epsilon << 1$ in (37) in which case

$$gh \approx \frac{\sigma}{F(\epsilon/2 - \sigma)} \tag{45}$$

and the gravity wave regime corresponds to the solutions of (37) where $\sigma >> \epsilon$ and

$$gh \approx (\sigma^2 - \epsilon\sigma - 1)/F. \tag{46}$$

(45) and (46) result in the following approximations for n

$$n_{RA} \approx \begin{cases} sgn(\sigma)\sqrt{SF}[(\epsilon/2 - \sigma)/\sigma]^{\frac{1}{2}} & |\sigma| \leq O(\epsilon) < O(1) \\ 0 & |\sigma| >> O(\epsilon) \end{cases} \tag{47}$$

$$n_{GA} \approx \begin{cases} sgn(\sigma)\sqrt{SF}(\sigma^2 - \epsilon\sigma - 1)^{-1/2} & |\sigma| \geq O(1) \\ 0 & |\sigma| << O(1) \end{cases} \tag{48}$$

n_{RA} and n_{GA} are approximated as

$$n_{RA} = \sqrt{SF}(1 - \hat{\sigma})/(\hat{r}_0 + \hat{r}_1\hat{\sigma} + \hat{r}_2\hat{\sigma}^2) \tag{49}$$

where $\hat{\sigma} = 2\sigma/\epsilon$ and

$$n_{GA} = \frac{\sqrt{SF}}{\sqrt{\eta}}\bar{\sigma}/(g_0\bar{\sigma}^2 + g_2) \tag{50}$$

where $\bar{\sigma} = (\sigma - \epsilon/2)/\eta$ where $\eta = \sqrt{1 + \epsilon^2/4}$. The coefficients $\hat{r}_0$,$\hat{r}_1$, $\hat{r}_2$,g_0 and g_2 are calculated by minimizing the respective reflection coefficients over the intervals of $.01 \leq \hat{\sigma} \leq .9$ and $.3 \leq \bar{\sigma}^{-1} \leq .99$. The total value of $n = n_{TA} = n_{RA} + n_{GA}$ is used in (44). Thus, n_{TA} is an approximation to (42) valid for gravity and Rossby waves in a form which allows conversion from wavenumber/frequency domain to wavenumber/time domain. The final equations in the wavenumber and time domain are of the form

$$\begin{aligned} &\frac{\partial\tilde{\xi}}{\partial z} - \tilde{\xi}/2 = \sqrt{SF}(\tilde{r}_1 + \tilde{r}_2 + \tilde{g}_1 + \tilde{g}_2) \\ &\frac{\partial}{\partial t}\tilde{g}_j = i(b + f\eta)\tilde{g}_j - f\bar{p}_j\tilde{\xi} \qquad \text{j} = 1,2 \\ &\frac{\partial}{\partial t}\tilde{r}_j = b(\hat{p}_j\tilde{\xi} - i\hat{x}_j\tilde{r}_j) \qquad \text{j} = 1,2 \end{aligned} \tag{51}$$

where $\tilde{\xi}$ represents any of the Fourier coefficients of $(\tilde{\chi}, \tilde{\psi}, \tilde{\phi}, \tilde{w})$, $\tilde{r}_j$ and $\hat{g}_j$ are new variables and $\hat{p}_j$, $\bar{p}_j$, $\hat{x}_j$ and $\bar{x}_j$ new constants. Tests using (51) indicate accurate solutions for both gravity and Rossby waves. A considerably more complex system would be required to accurately treat the full range of gravity, Rossby and Kelvin waves on a spherical earth. This paper of Rasch represents an excellent guide to future endeavors in this area.

4. LATERAL BOUNDARY CONDITIONS

In his review paper, Davies (1983) discusses four types of lateral boundary conditions commonly used by modellers. Following Davies, the schemes will be introduced in terms of the simple one–dimensional equation for the Riemann invariant,u, and characteristic speed,c.

i) Diffusive damping

$$u_t + cu_x = (\nu u_x)_x \quad (52)$$

which is applied over a boundary zone. This scheme must be uniformly applied to all variables to work effectively (e.g. Israeli and Orszag, 1981; Davies, 1983). Diffusive damping is a convenient approach to alleviate noise generated in the vicinity of the lateral boundary due to over specification or inappropriate boundary data (e.g. Burridge, 1975; Mesinger, 1977). The width of the boundary zone should be small compared to the wavelength of the basic system. Otherwise, significant damping and minor phase modification of incoming fields can occur. In models which use interactive nesting techniques (e.g. Clark and Farley, 1984) a zone of diffusive damping of two grid point width is used to absorb the smallest scale waves which are unresolved in the outer domain and as a result reflected at outflow boundaries.

ii) Tendency modification

$$u_t + cu_x = -\gamma(u - \bar{u})_t \quad (53)$$

which is applied over a boundary zone. The tendencies are assigned a weighted average of externally specified fields and internally determined fields such that the weighting associated with the external field varies from one at the boundary to zero at the inner extremity of the boundary zone (Kessel and Winninghoff, 1972; Perkey and Kreitzberg, 1976; Fritsch and Chappel, 1980; Maddox et al. 1981). In addition to the tendency modification the fields are also subjected to a spatial filtering procedure in the boundary zone. In this system $\bar{u} = \bar{u}(x,t)$ is prescribed and is itself a solution to the governing equations. It follows that the equation for $u' = u - \bar{u}$ takes the form

$$u'_t + c^* u'_x = 0 \quad (54)$$

where $c^* = c/(1+\gamma)$ and γ varies from zero in the interior to infinity at the boundary. The u' field is advected along at the modified speed c^* which reduces to zero at the boundary. Energy ,(u'^2), accumulates near the boundary and is extracted using spatial filters. However, as Davies(1983) points out the spatial filters are applied to u not u' which again means that the spatial filters must be applied over a narrow zone to not significantly damp the solution.

Davies (1983) presents an analysis of the reflection characteristics of the tendency modification scheme. The results of his analysis will be presented here. The approximation to (54) is taken as

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n) \tag{55}$$

where $\alpha = c^*\Delta t/\Delta x$ and $\alpha = \alpha_1$ in L_1 and $\alpha = \alpha_2$ in L_2. The solutions to (55) are assumed to take the form

$$\begin{aligned} &\text{in} \quad \mathrm{L_1}: \quad \mathrm{u_j} = \mathrm{e}^{-\mathrm{i}(\mathrm{k_1 x} - \omega \mathrm{t})} + \mathrm{r}(-1)^{\mathrm{j}} \mathrm{e}^{\mathrm{i}(\mathrm{kx} + \omega \mathrm{t})} \\ &\text{in} \quad \mathrm{L_2}: \quad \mathrm{u_j} = \mathrm{T}\mathrm{e}^{-\mathrm{i}(\mathrm{k_2 x} - \omega \mathrm{t})} \end{aligned} \tag{56}$$

where r and T are the reflection and transmission coefficients, respectively. From (55) we find

$$k_s \Delta x = \theta_s = sin^{-1}[\alpha_s^{-1} sin(\omega \Delta t)] \quad \text{for} \quad \mathrm{s} = 1, 2 \tag{57}$$

and in terms of θ_s

$$|r| = \frac{|sin\theta_1 - sin\theta_2|}{1 + cos(\theta_1 + \theta_2)}. \tag{58}$$

Figure 1 shows a reproduction of Fig. 3 from Davies (1983). In Fig. 1 (a) we see a graphical depiction of the solution to (57) and in 1(b) we see $|r|$ plotted against incident λ and $\Delta\lambda/\lambda$. It is interesting to note from this analysis that when $\alpha_2^{-1} sin(\omega\Delta t) > 1$ there is total reflection. The general reflection characteristics are shown in Fig. 1b.

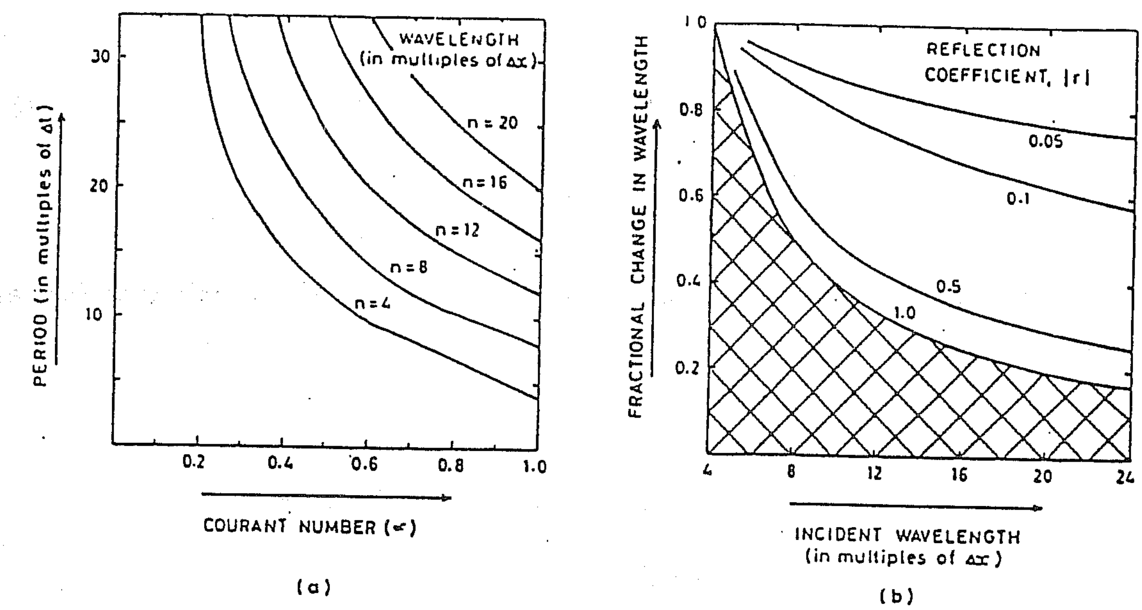

Fig. 1 Reflection characteristics (Davies, 1983) of the tendency modification scheme. (a) shows the relationship between Courant number, frequency and wavelength according to (57). (b) shows the reflection coefficient versus incident wavelength and fractional change in wavelength.

Reflection is certainly a short coming of this scheme. As a wave approaches the boundary γ increases and the wavelength decreases. This scheme then corresponds to changing the 'refractive index' of the computational system and can act to trigger a reflected wave. However, the $2\Delta x$ oscillatory nature of the reflected wave makes it particularly amenable to spatial filtering. In the case of application to cases with gravity waves one would have to take care to ensure that the refractive index changes due to the scheme do not adversely compete with those of the medium.

iii) Flow relaxation

$$u_t + cu_x = -K(x)(u - \bar{u}) \tag{59}$$

which is applied over a boundary zone (Davies,1976; Kallberg and Gibson, 1977a,b; Lepas et al., 1977; Gauntlett et al., 1978; Ninomiya and Tatsumi, 1980; Leslie et al., 1981; Richard et al., 1989). Davies (1983) presents analysis which indicates that to effectively damp an outgoing wave using a boundary zone with a small constant value of K, the zone must be excessively wide. An alternative to this design problem is to allow K to vary across the

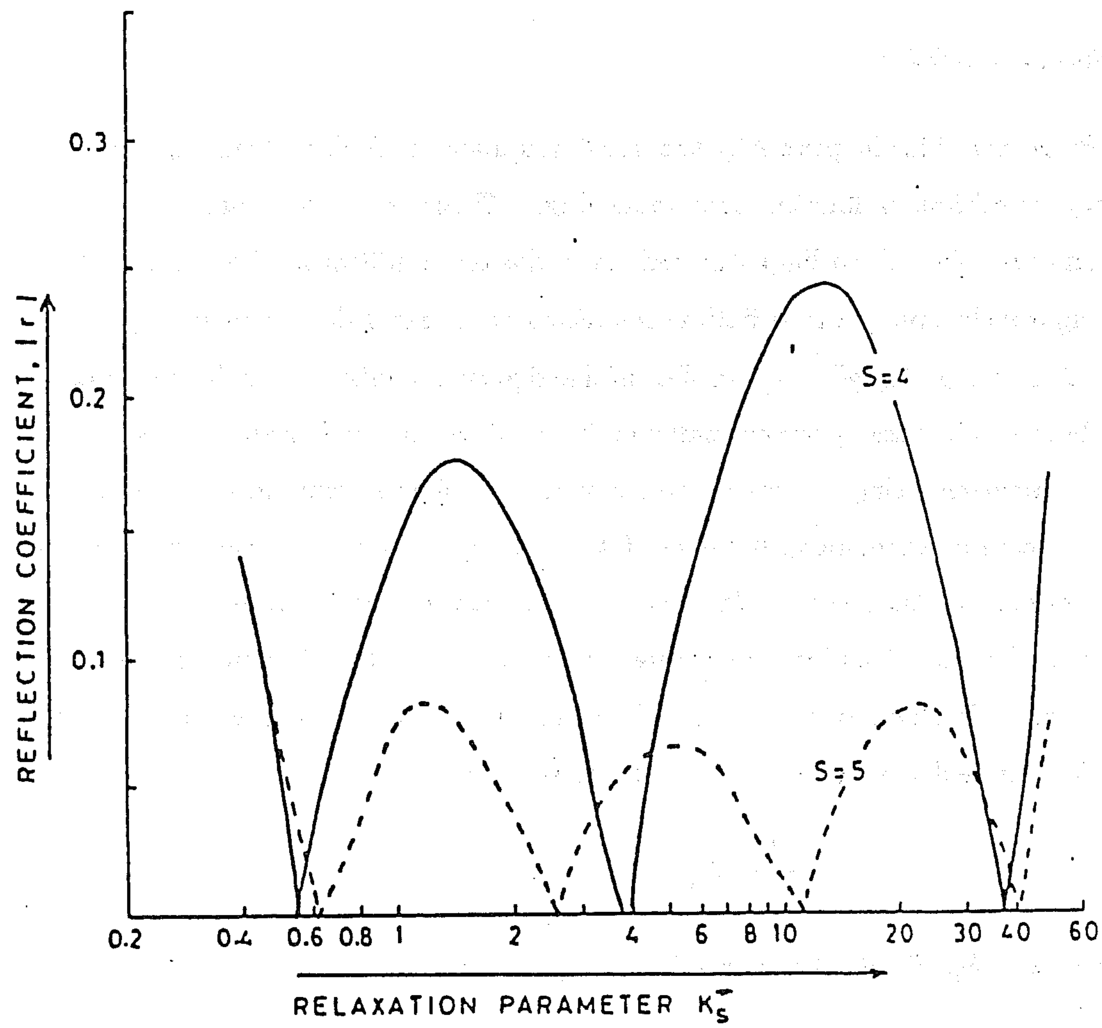

Fig. 2 Reflection coefficient (Davies, 1983) for the tuned flow relaxation scheme. $K_s^* = K_j \Delta x/c$ where K_j is the coefficient, Δx the grid size and c the phase speed.

boundary zone. The added degrees of freedom gained by permitting a spatial variation in K allows up to $(s-1)$ points of zero reflection for a boundary zone terminating at $j = s$ (K =0 for $j \leq 0$). He describes a procedure for tuning K_j^* (=$K_j \Delta x/c$) for the case of $\omega \Delta t << 1$. Values of K_j^* for $0 < j < s$ are determined in terms of K_s^*. Fig. 2 (Fig. 6 from Davies, 1983) shows the reflection coefficient versus K_s^*. Note the significant improvement going from $s = 4$ to $s = 5$. This particular analysis was for the scheme

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n) - K_j u_j^{n+1}. \tag{60}$$

Tatsumi (1980) has shown that adding a diffusion relaxation term can improve the scheme's performance.

iv) Radiation condition

Presently, this is probably the most popular method of treating the open boundary condition in limited area modelling. There is a wide range in level of approximation when describing the various radiation conditions. Béland and Warn (1975) rigorously apply the radiation condition to horizontally propagating gravity waves. Bennett (1976) gives examples of the rigorous treatments of inertia–gravity waves, barotropic Rossby waves and non–hydrostatic internal gravity waves. The rigorous approach using Fourier and Laplace transforms demands a tremendous amount of computer memory because of the non–local nature of the problem. As already discussed, the pseudo-differential operator technique of Enquist and Majda (1977) provides an attractive approximate method of avoiding the storage intensive history term. To demonstrate the non–local nature of the problem consider Béland and Warn's time–dependent linear Rossby wave equation

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\nabla_H^2\psi + \beta\frac{\partial}{\partial x}\psi = 0 \tag{61}$$

and let $\psi = \sum \phi_k e^{ikx}$ which reduces (61) to the form

$$\frac{\partial^2}{\partial y^2}\phi_k - k^2\left(\frac{\frac{\partial}{\partial t} + ikc_r}{\frac{\partial}{\partial t} + ik\bar{u}}\right)\phi_k = 0 \tag{62}$$

where $c_r = \bar{u} - \beta/k^2$ and if we apply Laplace transforms $\frac{\partial}{\partial t} \to p$ results in

$$\left(\frac{\partial}{\partial y} + k\sqrt{\frac{p + ikc_r}{p + ik\bar{u}}}\right)\left(\frac{\partial}{\partial y} - k\sqrt{\frac{p + ikc_r}{p + ik\bar{u}}}\right)\tilde{\phi}_k = 0. \tag{63}$$

The radiation condition is then

$$\frac{\partial}{\partial y}\tilde{\phi}_k + k\sqrt{\frac{p + ikc_r}{p + ik\bar{u}}}\tilde{\phi}_k = 0 \tag{64}$$

for $y \to \infty$. Using inverse Laplace transforms it is possible to show that (64) has the solution

$$\frac{\partial}{\partial y}\phi_k + k\phi_k = \int_0^t \phi_k(\tau)W_k(t-\tau)d\tau \tag{65}$$

where

$$W_k(t) = \frac{\beta}{2}exp[ik(\bar{u} - \frac{\beta}{2k^2})t]\left\{J_1(\frac{\beta t}{2k}) + iJ_0(\frac{\beta t}{2k})\right\}. \tag{66}$$

The right hand side of (65) is the history term which requires the storage of boundary data of ϕ_k for all time. Contrast the complexity of (65) against a first order Taylor series approximation of (64) using pseudo–differential operators which results in

$$(\frac{\partial}{\partial t} + ik\bar{u})(\frac{\partial}{\partial y}\phi_k + k\phi_k) = \frac{i\beta}{2}\phi_k. \tag{67}$$

Equation (67) assumes $\beta < 2k\sqrt{p^2 + k^2\bar{u}^2}$ and is, to my knowledge, an untested radiation condition.

The next level of approximation is perhaps that of Pearson (1974) where he applies a Sommerfeld condition to the vertical Fourier modes of the stream function, i.e.

$$\psi(x,z,t) = Re\sum \hat{\psi}_m(x,t)e^{imz} \tag{68}$$

and it is found that each vertical wavenumber m satisfies an equation of the form

$$\frac{\partial}{\partial t}\hat{\psi}_m + c_m\frac{\partial}{\partial x}\hat{\psi}_m = 0 \tag{69}$$

where $c_m = N/m$ for a non–rotating hydrostatic two–dimensional system. Adding a mean flow speed $\bar{u}$ to Pearson's system simply modifies $c_m \to c_m + \bar{u}$. However, adding rotational, non–hydrostatic or three–dimensional effects severly complicates the problem.

Klemp and Wilhelmson (1978) suggested a 'fixed phase speed' approach to treating the Sommerfeld condition

$$\frac{\partial\phi}{\partial t} + C_\phi\frac{\partial\phi}{\partial x} = 0 \tag{70}$$

Here they fix the phase velocity, C_ϕ, for the field ϕ to an over–estimate based upon the argument that the skewness of the reflection coefficient results in lower values for over–predictions than for under–predictions. This method does not consider any vertical or horizontal eigen function decomposition of ϕ. In testing this type of scheme, Clark (1979) and Hedley and Yau (1988) found significant variations in interior solutions due to run away circulation using different values for the phase velocity associated with (70).

Orlanski (1976) also assumed a Sommerfeld radiation condition of the form (70) holds for the various field variables at the x=constant boundary. The phase velocity, C_ϕ, is locally calculated without performing any vertical or horizontal eigen mode decomposition as in, say, the scheme of Pearson(1974). Analytically, then, this scheme is likely to be exact only in the idealized case of two-dimensional, hydrostatic, non-rotating flow for the case of a linear mean flow plus a single wave component. Comparing with Pearson's scheme the exact answer would be $C_\phi = \bar{u} \pm N/m$ providing the phase is directed out of the domain. Numerically, Orlanski approximated (70) using the centered in time and space approximations

$$\phi_j^{n+1} = \frac{[1-(\Delta t/\Delta x)C_\phi]}{[1+(\Delta t/\Delta x)C_\phi]}\phi_j^{n-1} + \frac{2(\Delta t/\Delta x)C_\phi}{[1+(\Delta t/\Delta x)C_\phi]}\phi_{j-1}^n \tag{71}$$

where

$$C_\phi = -\frac{[\phi_{j-1}^n - \phi_{j-1}^{n-2}]}{[\phi_{j-1}^n + \phi_{j-1}^{n-2} - \phi_{j-2}^{n-1}]}\frac{\Delta x}{2\Delta t} \tag{72}$$

which are designed to minimize the excitation of computational modes. An erroneous analysis was presented by Orlanski (1976) which claimed to show that (71) and (72) resulted in an exact numerical solution for the idealized case described above. These equations are in fact nonlinear and one has to take care when keeping track of the real and imaginary parts of ϕ. See appendix for an evaluation of the reflection coefficient for the general Orlanski type schemes. As shown in Miller and Thorpe (1981) all the numerical schemes they tested of the above type had significant reflection coefficients.

Miller and Thorpe (1981) present an interesting comparison between the Orlanski(1976) leap-frog formulation of the pseudo-radiation condition and an upstream formulation which they formulate as

$$\phi_j^{n+1} = \phi_j^n(1-\hat{\epsilon}_U) + \hat{\epsilon}_U\phi_{j-1}^n \tag{73}$$

for the upstream formulation and

$$\phi_j^{n+1} = (\frac{1-\hat{\epsilon}_L}{1+\hat{\epsilon}_L})\phi_j^{n-1} + (\frac{2\hat{\epsilon}_L}{1+\hat{\epsilon}_L})\phi_{j-1}^n \tag{74}$$

for the leap frog formulation where $\hat{\epsilon}_U$ and $\hat{\epsilon}_L$ are the extrapolation coefficients for the upstream and leap–frog formulations, respectively. They present analyses on 10 different schemes using a combination of temporal and spatial derivatives. They also present cases eliminating the temporal derivative by assuming a single wave solution. Their results indicate that the radiation conditions give the lowest order truncation error.

Hedley and Yau (1988) present a two–dimensional contour plot of C_ϕ using (72) which shows a significant level of noise. Figure 3 shows a reproduction of their Fig. 12. Raymond and Kuo (1984) show that the range in calculation of C_ϕ using the Orlanski approach is between plus and minus infinity as a direct result of substituting ϕ which is multi-dimensional in a formula that only allows for advection in one dimension. They introduce a condition suitable for multi-dimensional flow based upon

$$\frac{\partial \phi}{\partial t} + C_x \frac{\partial \phi}{\partial x} + C_y \frac{\partial \phi}{\partial y} = 0. \tag{75}$$

and describe two approaches to finding C_x and C_y. One is based upon extrapolation and the other upon using the governing equation. On a topic related to the dimensionality issue, Bennett and Kloeden (1978) point out that flow tangential to the boundary is a cause of severe boundary problems.

Raymond and Kuo found some significant improvements using the multi—dimensional formulation of C_x and C_y. This was particularly true for the constant advection and barotropic vorticity equation. Their results for a cold front case were not as convincing due to the lack of multi-dimensionality of that problem (by Hedley and Yau, 1988).

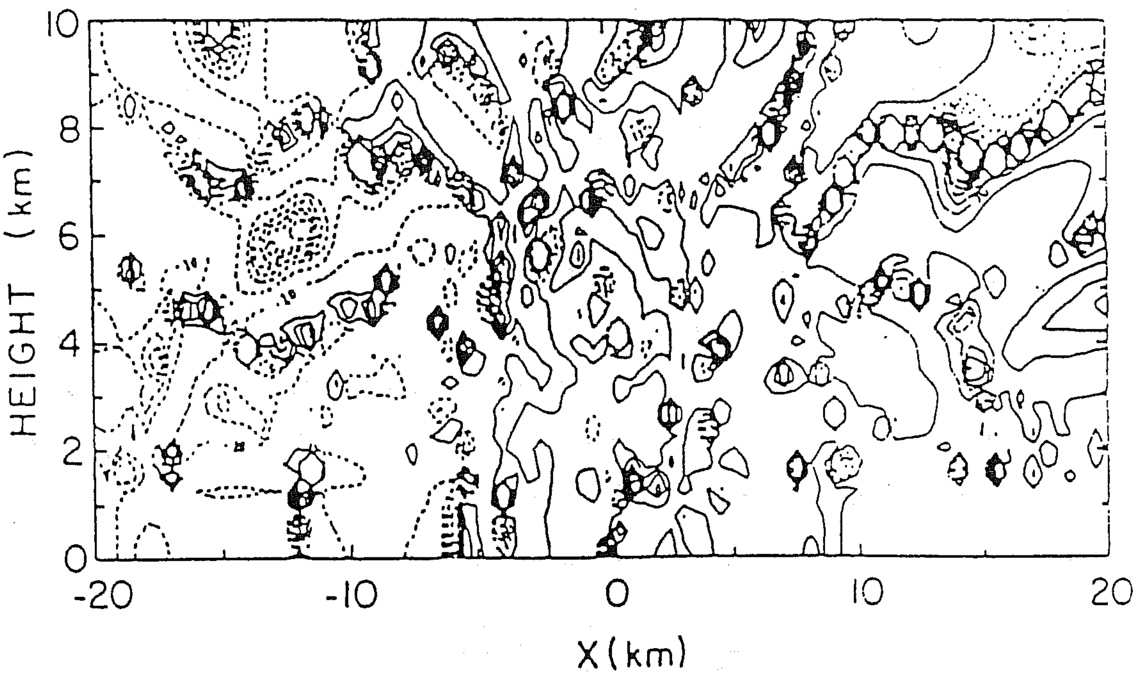

Fig. 3 Horizontal phase speeds, C_ϕ, from the cloud simulation of Hedley and Yau (1988). Contour interval is 10 m s^{-1}. Dashed lines indicate negative values.

5. SOME FURTHER ADJUSTMENT PROCEDURES

i) Relaxation of tangential velocity components and potential temperature at Inflow Boundaries

When simulating flow with a limited area model there are a number of boundary condition problems that occur which are due to internal generation of waves of various types. This is particularly a problem for calculations bounded by orography. Some annoying examples are boundary conditions leading to the anomalous production of vorticity and/or an updraught or downdraught which clings to the inflow 'wall'. In order to allow information to enter the integration domain we desire a specification of the tangential velocity components and θ at inflow boundaries. A direct specification using external data or some prognostic procedure can lead to sharp gradients of these variables across the boundary so a relaxation to specification is sometimes used to alleviate this problem.

One relaxation scheme used by Kurihara and Bender (1983) is to relax u and v towards their prescribed reference values,

$$u_{j=1}^{n+1} = (1-\epsilon)u_{j=1}^{*} + \epsilon U_e \quad (76)$$

where u_j^* is a first guess of $u_{j=1}$ using some open boundary condition. They apply a similar approach to the normal velocity components at all boundaries which is also the scheme used by Cho and Clark (1981). Applications of this scheme to large scale waves shows some improvement in the results.

Recently for our orographic flow simulations we found it necessary to adopt a version of (76) based upon the gradient of the field rather than upon the field itself. The approach adopted is to apply

$$u_{j=1} = (1-\epsilon)u_{j=2} + \epsilon U_e \quad (77)$$

$$\theta_{j=1} = (1-\epsilon)\theta_{j=2} + \epsilon \Theta_e \quad (78)$$

during each time step of the model. Similar treatments are used at the other inflow boundaries for the tangential velocities and θ. The rationale behind (77) and (78) is to attempt to keep $\frac{\partial u}{\partial y}$ and $\frac{\partial \theta}{\partial y}$ as close to zero as possible but still allow new information to be introduced into the domain. Detailed analysis of this scheme has yet to be performed. Figure 4 shows two examples of simulated flow over the Colorado Rockies. Figure 4(a) used $\epsilon = 1$ in (77) whereas Fig. 4(b) used $\epsilon = .025$.

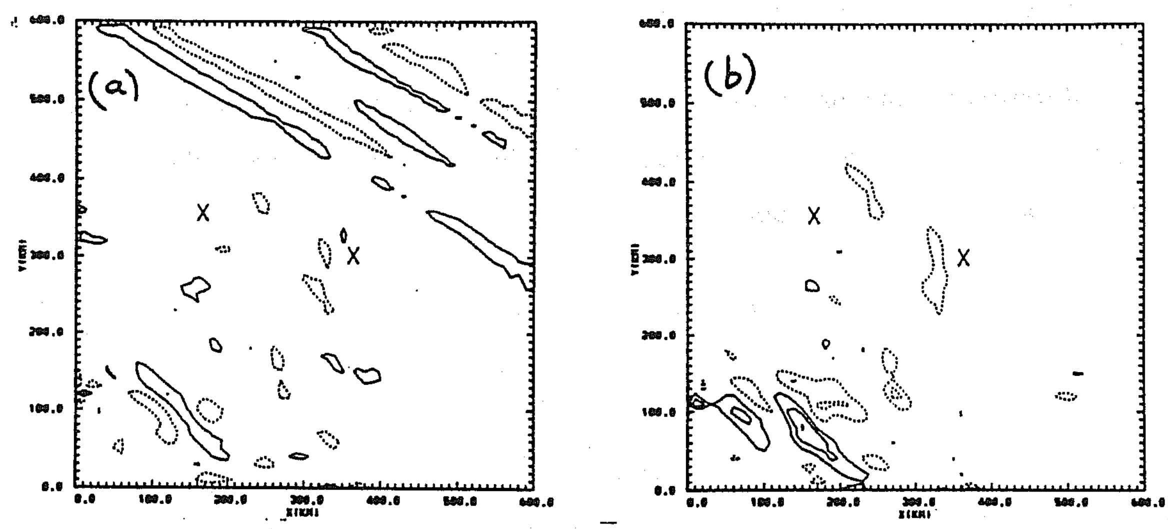

Fig. 4 Plots of vertical vorticity at z =8 km above a ground reference level for flow over the Colorado Rockies. (a) shows results using (77) with ϵ = 1.0 and (b) shows results using ϵ =1.

Figure 5 shows examples where (78) was applied to θ using $\epsilon = 1$ (Fig. 5a) and where is was applied using $\epsilon = .025$ (Fig. 5b) for the simulation of stable air flow over north central Arizona. There is a dramatic improvement in the results when (78) is applied. The anomalous narrow inflow region of w in Fig. 5(a) is caused by gravity waves, excited in the interior of the domain, propagating upstream towards the 'inflow' boundary. The environmental specification, Θ_e contains no gravity wave information. As a result a sharp gradient in θ across the boundary develops which can generate either an updraught or down draught.

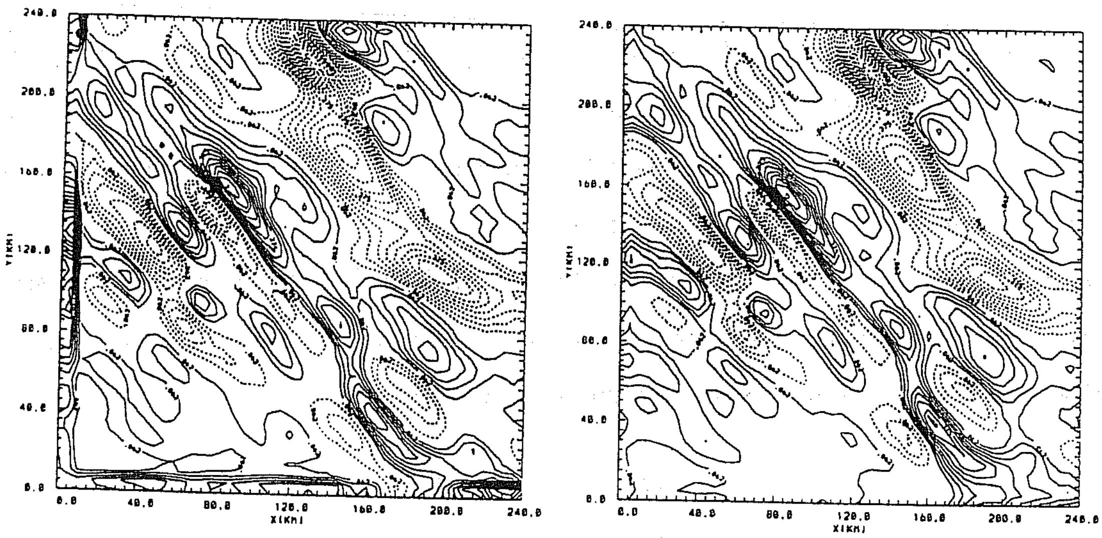

Fig. 5 Plots of w at z = 4 km above a surface reference level for flow over the Mogollon rim area of Arizona. (a) shows results using (78) with ϵ = 1 and (b) shows results with ϵ = .025.

ii) Relaxation of normal velocity components at inflow boundaries

The pseudo–radiation condition such as (71) treats the issue of specification in a rather cavalier manner. Unlike flow relaxation or tendency modification there is no assurance that environmental (or reference state) information will be able to appropriately affect the solution. In fact, the solution can systematically drift away from the reference state (e.g. Clark, 1979; Hedley and Bender, 1988). To alleviate this problem Cho and Clark (1981) apply a time relaxation to the pseudo–radiation condition predictions of the normal velocity components at inflow boundaries whereas Kurihara and Bender (1983) apply it at all boundaries. The equation used is

$$u_j^{n+1} = (1-\epsilon)u_j^* + \epsilon U_e(t) \qquad (79)$$

where u_j^* is the normal component of u predicted using, say, (71). Use of (79) has been found to alleviate such problems as runaway circulation. Furthermore it allows the boundary flow to adjust with time to the reference state.

iii) Mass balance adjustments required for $\vec{V}_n$

Once future values of $\vec{V}_n$ have been determined for all boundary points using a combination of specification and extrapolation procedures from the interior one still has to insure that these predictions satisfy a mass balance. This is absolutely necessary within the anelastic system of equations since if $\nabla \cdot \rho\vec{V} = 0$ is not satisfied in a global sense then no solution to p exists. In other models an imbalance could be a strong source of acoustic energy. A couple of approaches have been used to adjust $\vec{V}_n$. The first and not a particularly advisable one is to simply let

$$\rho\vec{V}_n = (1+\alpha)\rho\vec{V}_n^* \tag{80}$$

where $\vec{V}_n^*$ is the initial unadjusted value. α is easily calculated using a crude least squares approach such that the numerical approximation of

$$\oint_A \rho\vec{V}_n \cdot dA = 0 \tag{81}$$

is satisfied.

The disadvantage of using (80) is that the adjustment has the same vertical structure as the normal velocity itself. In the case of such problems as runaway circulation this type of adjustment can aggrevate the situation (Redelsperger; personal communication). (80) was infact used in Clark (1979). A probably better approach is to let

$$\rho\vec{V}_n = \rho\vec{V}_n^* \pm \alpha\rho \tag{82}$$

where the signs are chosen to give the required convergence or divergence. The amount of convergence or divergence in the x- and y- direction can be divided according to the mean flow amplitude in those direction. This is the type of adjustment currently used in the Clark model and by Hedley and Yau (1988) who discuss this topic to some length.

6. CONCLUSIONS

This paper has only briefly touched on the various issues involved in formulating the various methods being used by limited area modellers to prescribe their lateral and upper boundary conditions. From the sample of the papers covered it seems clear that there is still a lot of room for improvement in the treatment of lateral boundary conditions for limited area modelling and upper boundary conditions for meteorological models in general. The upper boundary condition treatments appear to be in much better shape than do the lateral boundary conditions. If all else fails, for example, we have recourse to the upper level absorber which can with nesting be made reasonably cost effective. The lateral boundary condition treatments are still at a rather pragmatic level using a combination of physics, technology and art to achieve results. I refer the interested reader to the reviews in the papers of Davies (1983), Miller and Thorpe (1981), Rasch (1986) and Hedley and Yau (1988) for further reading on this challenging subject.

APPENDIX

The reflection coefficient of the centered in time and space scheme

$$\phi_j^{n+1} = \phi_j^{n-1} - \alpha(\phi_{j+1}^n - \phi_{j-1}^n) \qquad (A-1)$$

with the boundary condition

$$\phi_j^{n+1} = \Big(\frac{(1-\epsilon)}{(1+\epsilon)}\phi_j^{n-1} + \frac{2\epsilon}{1+\epsilon}\phi_{j-1}^n \qquad (A-2)$$

is desired. ϕ_j^n has the four possible solutions

$$e^{\pm i(kx-\omega t)} \;\; ; \;\; (-1)^j e^{\pm i(kx+\omega t)} \;\; \text{for} \;\; \alpha > 0$$

and

$$e^{\pm i(kx+\omega t)} \;\; ; \;\; (-1)^j e^{\pm i(kx-\omega t)} \;\; \text{for} \;\; \alpha < 0 \qquad (A-3)$$

where there are the two physical and two computation modes. In all cases (A-1) gives $\sin(\omega\Delta t) = \alpha sin(k\Delta x)$. To find the reflection coefficient we let

$$\phi_j^n = e^{i(kx-\omega t)} + r(-1)^j e^{-i(kx+\omega t)} \qquad (A-4)$$

and consider the case of $\alpha > 0$. Substituting (A-4) into (A-2) and defining $x = 0$ as the boundary point results in

$$r = \frac{-cos(\omega\Delta t) + cos(k\Delta x) + isin(k\Delta x)(\alpha/\epsilon - 1)}{cos(\omega\Delta t) + cos(k\Delta x) - isin(k\Delta x)(\alpha/\epsilon - 1)} \qquad (A-5)$$

$r = 0$ only for the case of $\epsilon = \alpha = 1$, otherwise there is reflection for the pure single wave case. In the limit

$$r \rightarrow \frac{ik\Delta x}{2}(\alpha/\epsilon - 1) \text{ as } \omega\Delta t \rightarrow 0. \qquad (A-6)$$

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