FINITE DIFFERENCE METHODS

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Summary: An overview of finite difference methods for atmospheric models is given. Preservation of important properties of the continuous equations is discussed.

1. INTRODUCTION

Though the spectral method is used for most atmospheric models used for weather forecasting and climate studies, the finite difference method is still used, for instance by the U.K.Meteorological Office unified forecast/climate model and the GLA climate model. The finite difference method is also used for most limited area models, and is also used for the vertical discretization in most global spectral models. The design of finite difference methods for atmospheric models now follows well-established principles. This paper gives an overview of the main considerations followed, other papers in this volume take up particular aspects in greater detail. The basic theory of finite difference methods applied to the atmospheric equations is set out in much greater detial in Mesinger and Arakawa (1976) and Haltiner and Williams (1981).

2. PRINCIPLES OF THE FINITE DIFFERENCE METHOD

In order to apply the finite difference method, the equations must be averaged over a horizontal and vertical scale rather larger than the gridlength to be used. Additional terms (the subgrid model) then appear on the right hand side of the equations. These typically take the form of smoothing or filtering operators, but more elaborate subgrid models are used in mesoscale models. The effect of the averaging is to make the solution smooth over the grid scale. This is why the subgrid model primarily looks like a smoothing operator. Since the solution is smooth, the derivatives in the equations can now be approximated by finite differences. It should be noted that, since there is no way of rigorously deriving the subgrid model, there is a loss of predictability associated

with having to choose it. This may be particularly important where there are organised structures such as fronts on scales smaller than the averaging distance.

In most applications of finite difference methods, the grid point values represent cell averages. The finite difference equations are then derived by integrating the equations over the cell. When analysing the accuracy of the methods, however, it is usual to substitute a particular solution of the exact equation into the finite difference equation, the residual remaining is the truncation error. An nth order finite difference scheme will solve equations containing derivatives up to order m exactly for polynomials up to order n+m-1. Alternatively, the error will be $O(k^n)$ for a trigonometric function $A\cos kx + B\sin kx$. This can easily be seen by expanding the trigonometric functions in their polynomial series, and calculating the first error term in the approximation to the mth derivative. This can be illustrated by considering the finite difference approximation to $\partial C/\partial x$:

$$\frac{1}{2\Delta x}(C(x+\Delta x)-C(x-\Delta x)).$$

Substituting C=Acos(kx) gives

$$\partial C/\partial x \sim A(\cos(k(x+\Delta x))-\cos(k(x-\Delta x)))$$
$$=-(A/\Delta x)\sin(kx)\sin(k\Delta x)$$
$$=-kA\sin(kx)(1-k^2\Delta x^2/6+O(k\Delta x)^4)$$

showing that the scheme is second order accurate.

3. STRUCTURE OF THE EQUATIONS RELEVANT TO FINITE DIFFERENCE METHODS

3.1 Basic equations

In order to obtain an accurate representation of the evolution of the atmosphere, certain properties of the continuous equation have to be reproduced exactly in their finite difference analogue. The various important subsystems within the equations of motion may require differing numerical treatments, both to ensure computational economy, and to obtain accuracy. Many of the considerations apply equally to spectral methods, and have been discussed in the previous two papers in this volume.

In order to illustrate these points in the simplest way, we use the equations in Cartesian coordinates in the horizontal and pressure

coordinates in the vertical. Mesinger's paper in this volume treats the issue of the vertical coordinate and its effect on the finite differencing error at length.

The equations, in standard notation, are then

$$\frac{Du}{Dt} + \frac{\partial \phi}{\partial x} - fv = F_u \tag{1}$$

$$\frac{Dv}{Dt} + \frac{\partial \phi}{\partial y} + fu = F_v \tag{2}$$

$$\theta = \theta_0(p) + \theta' \tag{3}$$

$$\frac{D\theta'}{Dt} + \omega \frac{\partial \theta}{\partial p}_0 = F_\theta \tag{4}$$

$$\frac{Dq}{Dt} = F_q \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{6}$$

$$\pi = (p/p_0)^\kappa \tag{7}$$

$$\frac{\partial \phi}{\partial \pi} = c_p(\theta' + \theta_0) \tag{8}$$

Note that θ is expressed as the deviation from a reference state θ_0 independent of horizontal position. π is often referred to as the Exner function, $\kappa = R/c_p$. The right hand side terms include forcing terms and the subgrid model. The boundary conditions are

$$\omega = 0 \text{ at } p=0. \tag{9}$$

$$\omega = \frac{Dp_s}{Dt} \text{ at } p=p_s, \text{ the Earth's surface.} \tag{10}$$

$$\phi = \phi_s \text{ at the Earth's surface.} \tag{11}$$

3.2 Gravity waves and geostrophic adjustment

Linearising the D/Dt terms and neglecting the right hand side terms gives a set of equations for gravity-inertia waves. These are oscillations about a state of rest in geostrophic and hydrostatic balance given by

$$\frac{\partial \phi}{\partial x} - fv = 0 \tag{12}$$

$$\frac{\partial \phi}{\partial y} + fv = 0 \tag{13}$$

$$\frac{\partial \phi}{\partial \pi} = c_p(\theta' + \theta_0) \tag{14}$$

The equations for the oscillations can be condensed into a single elliptic

equation for ω, which can be derived by differentiating (1) by x, p and t, (2) by y, p and t, and substituting ϕ by θ using (8) and $\partial\theta/\partial t$ using (4). Equation (6) is then used to replace $(\partial u/\partial x+\partial v/\partial y)$ by $-\partial\omega/\partial p$. This gives

$$\frac{\partial^2}{\partial\pi\partial p}\left(\frac{\partial^2\omega}{\partial t^2}+f^2\omega\right)+\nabla^2\left(\omega\frac{\partial\theta}{\partial p}\right)-f\frac{\partial}{\partial\pi}(\beta v)=F \qquad (15)$$

The right hand side term is derived from the forcing terms. The complete equations can be written this way if the nonlinear terms are included in F. The reaction of the equations to the forcing terms thus consists of an oscillation in time (given by the $\partial^2/\partial t^2$ term) about a slowly varying solution where ω is determined by the second and third terms. The nature of the solution depends on the relative size of these terms. If the horizontal scale is greater than the Rossby radius of deformation NH/f, where N is the Brunt-Vaisala frequency and H the scale height, then the second term is dominant and the wind fields are largely determined by the evolution of the pressure and temperature fields. If the horizontal scale is less than the Rossby radius, then the converse applies.

These oscillations have phase speeds up to $300ms^{-1}$. They therefore have to be treated either by using explicit finite difference methods in time, with a shorter timestep than that which could be used for the rest of the model, or by using implicit methods as in spectral models (see paper by Machenhauer in this volume). If implicit methods are used, then an equation of the form (15) has to be solved for ω at the new time level.

3.3 Advection and conservation

The D/Dt terms in the equations represent transport of values from their position at a previous time level along a trajectory to a position at a new time level. This means that the new values are a rearrangement of the old, which means in particular that the integral over the whole volume of any function of the rearranged quantity is preserved. It is not possible to imitate this condition exactly in finite difference methods, but it is often possible to ensure that the integral of the quantity and of the square of the quantity are exactly conserved. This is not sufficient to guarantee good behaviour of the solution, but can reduce the risk of instability. The rearrangement condition also means that the values of the transported quantity are bounded by the initial values. This property is enforced in shape-preserving advection schemes, see the paper by

Smolarkiewicz in this volume. The transport property is used directly in semi-Lagrangian methods, described in the paper by McDonald in this volume.

Eulerian finite differnce methods have to write the advection terms in the expanded form

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + \omega\frac{\partial\theta}{\partial p} = 0. \qquad (16)$$

The continuity equation (6) can be used to write this in flux form

$$\frac{\partial\theta}{\partial t} + \frac{\partial(u\theta)}{\partial x} + \frac{\partial(v\theta)}{\partial y} + \frac{\partial(\omega\theta)}{\partial p} = 0. \qquad (17)$$

The terms involving spatial derivatives are now the divergence of a flux vector. Integrating (17) over the whole domain shows that the integral of θ with respect to x,y and p is conserved under advection. This is a requirement for climate integrations, where the total heat and moisture budget has to be accurately represented, and for other integrations (such as pollution modelling) where the total amount of an advected substance is significant. The natural finite difference approximation to (17) is obtained by integrating it over a grid volume, and then replacing the integral of the flux divergence by the integral of the normal flux

$$(u\theta, v\theta, \omega\theta).\mathbf{n} \qquad (18)$$

round the boundary of the grid box, where **n** is the normal vector to the boundary.

3.4 Angular momentum and energy integrals

In extended range and climate integrations, it is important that the total angular momentum is only changed by a surface torque, and that the conversions between kinetic and potential energy sum to zero when integrated over the whole domain. It should be noted that the dynamical equations set out above do not conserve energy because of the energy sink in the sub-grid model. In climate integrations it is necessary to diagnose this sink and return the energy to the model as heat.

Enforcement of the angular momentum principle requires that the integral of the pressure gradient term $(\partial\phi/\partial x, \partial\phi/\partial y)$ can be written as a boundary contribution from the Earth's surface only. This is trivial in the form of equations used to illustrate this paper, but requires careful treatment of

the vertical differencing in the hydrostatic equation (8) if terrain-following coordinates are used, see the paper by Mesinger in this volume.

The conversion from kinetic to potential energy can be found by multiplying (1) and (2) by u and v respectively. Then

$$\frac{D}{Dt}\left(\frac{1}{2}(u^2+v^2)\right) + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} = uF_u + vF_v \qquad (19)$$

The left hand side terms are the rate of change of kinetic energy following a parcel and the pressure work term. Integrating the pressure work term over the whole volume and integrating by parts gives

$$\int -\phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \, dxdydp$$

Using the continuity equation (6) and integrating by parts again gives $\int -\omega\frac{\partial\phi}{\partial p}dxdydp$, which can be written using the hydrostatic equation (8) and converting from θ to T as $\int -\frac{RT\omega}{p}\,dxdydp$.

Correct treatment of these conversions in a finite difference model means that all these manipulations have to be carried through in finite differences, which implies consistency between the finite difference treatments of all the basic equations, see Arakawa and Suarez (1983).

4. FINITE DIFFERENCE SCHEMES FOR THE ADJUSTMENT TERMS

4.1 Arrangement of the variables on the grid

The accuracy of the adjustment scheme depends to a large extent on how well the ω equation (15) is represented. This can be determined by combining the finite difference representations of (1) to (8) into a finite difference approximation to (15). This topic is treated in detail by Mesinger and Arakawa (1976), so that the results will only be summarised in this paper.

The main consideration in the choice of schemes is the arrangement of variables on the grid. It is almost universal to divide the atmosphere into layers, with the Earth's surface as a layer boundary, and the 'top' of the atmosphere as a layer boundary. The horizontal velocity components are held at midpoints of layers and the vertical velocity is held on layer

boundaries. In order to solve equations (1) and (2), the geopotential ϕ must be calculated at the midpoints of the layers. This requires vertical integration of (8). There are two methods. If the potential temperature θ is held at layer midpoints (the Lorenz grid), then the lower boundary condition (11) is applied at the Earth's surface, and (8) is integrated over complete layers

$$\phi_{k+1}-\phi_k = c_p\theta_{k+1/2}\delta\pi \tag{20}$$

The final integration to the layer midpoint k+3/2 takes the form

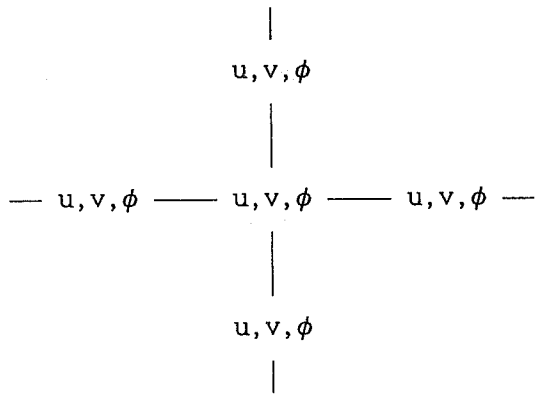
$$\phi_{k+3/2}-\phi_{k+1} = \alpha c_p\theta_{k+3/2}\delta\pi \tag{21}$$

The factor α is chosen to ensure that the energy and angular momentum properties set out in section 3.4 are preserved, and depends on the vertical coordinate system being used.

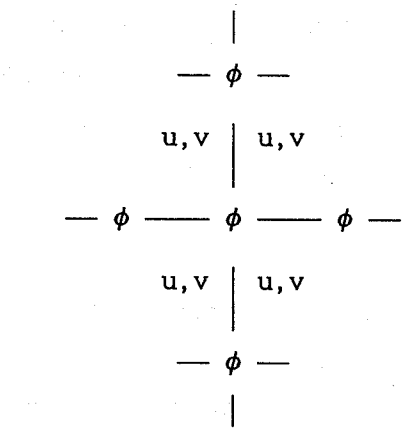
In the second method (the Charney-Phillips grid) the potential temperature is held at layer boundaries. After an initial integration from the Earth's surface to the midpoint of the first layer, the integration then proceeds directly from the midpoint of one layer to the next. It is harder to make the manipulations in section 3.4 work when this grid is used.

The various arrangements of variables in the horizontal were classified by Arakawa as follows. In each case the vertical velocity ω is held at the same position as the potential temperature θ and geopotential ϕ.

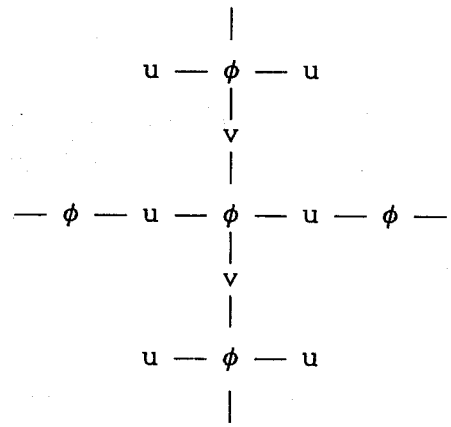
A grid



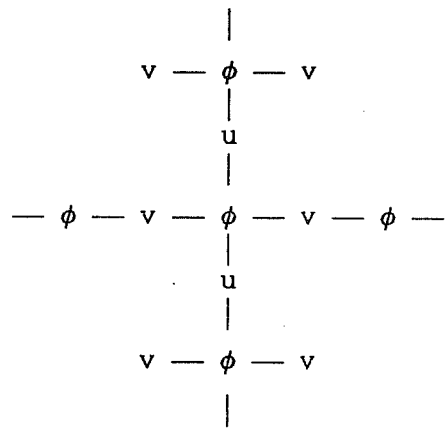
B grid



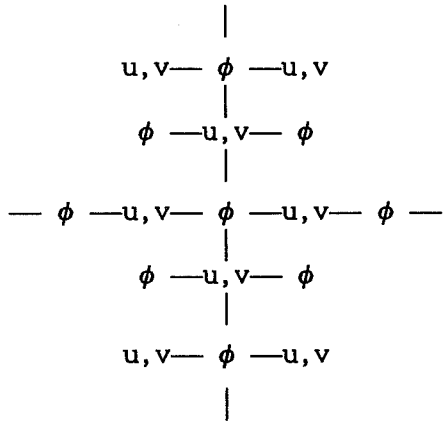
C grid



D grid



E grid



The terms in the adjustment equations are approximated by central differences, with averaging if necessary to produce the result in the correct position on the grid. Thus the approximation to the term $\partial\phi/\partial x$ in equation (1) is

A grid: $\delta_{2x}\phi$

B grid: $\overline{\delta_x\phi}^{y}$

C grid: $\delta_x\phi$

D grid: $\overline{\delta_{2x}\phi}^{y}$

E grid: $\delta_x\phi$

The standard finite difference notation $\delta_x\phi = (\phi(x+\frac{1}{2}\delta x)-\phi(x-\frac{1}{2}\delta x))/\delta x$ and $\bar{\phi}^{x}=\frac{1}{2}(\phi(x+\frac{1}{2}\delta x)+\phi(x-\frac{1}{2}\delta x))$ is used.

Making similar approximations to the other terms in the adjustment equations and combining them into an equation of the form (15) gives and approximation to the $\partial^2\omega/\partial t^2$ term of the form

$$\left(\partial^2\omega/\partial t^2_{k+1} - \partial^2\omega/\partial t^2_{k}\right)/\Delta p_{k+1/2} - \left(\partial^2\omega/\partial t^2_{k} - \partial^2\omega/\partial t^2_{k-1}\right)/\Delta p_{k-1/2}$$

The approximation to the $f^2\omega$ term takes a similar form in the vertical. However, it contains additional horizontal averaging on the C and D grids, where the velocity components are held in different places. The approximation to the ∇^2 term involves the following horizontal

approximations to $\nabla^2\omega$:

A grid: $\delta_{2x}(\delta_{2x}\omega) + \delta_{2y}(\delta_{2y}\omega)$

B grid: $\delta_{x}(\delta_{x}\omega) + \delta_{y}(\delta_{y}\omega)$

C grid: $\delta_{x}\delta_{x}\omega + \delta_{y}\delta_{y}\omega$

D grid: $\delta_{2x}(\delta_{2x}\omega) + \delta_{2y}(\delta_{2y}\omega)$

E grid: $\delta_{x}\delta_{x}\omega + \delta_{y}\delta_{y}\omega$

In comparing these expressions, it should be noted that the effective gridlength on the E grid is $\sqrt{2}$ times larger than on the other grids. The most compact and therefore most accurate stencil is that on the C grid. Stencils including averaging may help to reduce noise with explicit time integration schemes. However, with implicit time integration schemes, the stencil appears on the left hand side of the elliptic equation to be solved, and averaging then is a disadvantage as it makes the problem ill conditioned. The overall performance of the schemes depends on the relative sizes of the second and third terms in (15). The B,C and E grids all have advantages under the right conditions, while the A and D grids are less accurate. The D grid gives trhe most accurate representation of the geostrophic balance itself, but this is of no advantage in solving (1) to (8) as the approximations to (1) and (2) have to be generated at u and v points. However, if a balanced model was being integrated and equation (15) solved without the time derivative term, then the D grid would be advantageous.

The vertical stencil is most compact on the Charney-Phillips grid, where the third term in (15) contains no vertical averaging. On the Lorenz grid the third term is averaged over three levels of ω.

4.2 Time integration schemes for adjustment

The standard schemes are summarised below. The prognostic equation for surface pressure present in the most common vertical coordinate formulations is treated in the same way as the potential temperature equation (4).

Leapfrog. The time derivatives in equations (1), (2) and (4) are approximated by

$$\frac{\partial u}{\partial t} \sim \delta_{2t} u.$$

This method requires three time levels of data, and special measures have to be taken to prevent separation of the values at odd and even time levels. The maximum timestep Δt is $\Delta x/c$ on the A grid, but $\Delta x/2c$ on the C grid, because the shortest resolvable wavelengths are treated more accurately.

Forward backward. The time derivative in equation (1) is approximated by a forward difference

$$\frac{\partial u}{\partial t} \sim \delta_t u(t+\tfrac{1}{2}\Delta t).$$

The right hand side of the equation is held at time t. The time derivative in equation (2) is similarly approximated, using the value of the pressure gradient at time t, but the value of u at time $t+\Delta t$. Equation (4) is solved using values of the right hand side at time $t+\Delta t$, noting that the basic state θ_0 is time independent. This method only requires two time levels of data. The maximum timestep is $2\Delta x/c$ on the A grid and $\Delta x/c$ on the C grid.

Implicit. The right hand sides of all equations are time averaged:

$$F_u \sim (\alpha F_u^{t+\Delta t} + (1-\alpha) F_u^t).$$

The value $\alpha=1/2$ gives a scheme with no amplification or damping, similar to the leapfrog and forward-backward schemes, and no timestep restriction. Values $\alpha>1/2$ give damping, which can be useful under some circumstances. The right hand sides of the equations include unknown values, and therefore an elliptic equation of the form (15) has to be solved.

Semi-implicit methods. These are implemented in a similar way to spectral models, see the paper by Machenhauer in this volume. Only the linearised parts of equations (1), (2) and (4) are treated implicitly. The remaining terms are treated explicitly with a longer timestep. The splitting procedure is only stable if the internal gravity wave speeds of the linearised equations in finite difference form are greater than those of

the nonlinear equation, Simmons, Hoskins and Burridge (1978).

Split-explicit methods. The most common version is to use the forward-backward method on the B or E grid for the linearised version of equations (1) and (2), and for the vertical advection in equation (4). Several timesteps are performed before the remainder of the equations are included with a longer timestep. If storage is available, the time truncation errors can be substantially reduced by saving the increments from the long timesteps and adding fractions of them in at each short timestep, Purser and Leslie (1991). Since there is no spatial averaging in the Coriolis terms on these grids, the Coriolis terms can easily be treated implicitly. A conservative version suitable for climate modelling can be obtained by using the linearised version of equation (4). the same restrictions on the choice of θ_0 apply as for semi-implicit methods. The average velocity from the adjustment steps is saved and used in the subsequent calculation of the advection terms.

5. ADVECTION SCHEMES

5.1 Time integration

Advection schemes are treated comprehensively in the papers by Smolarkiewicz and McDonald in this volume. This section gives a brief introduction to finite difference advection schemes on a fixed grid.

If the advecting velocity is uniform, exact finite difference schemes exist which simply transport data by one gridlength in one timestep. However, in atmospheric models the advecting velocity is very variable. If explicit time integration is used, as is normal, then the timestep will be limited by the highest speed, and over most of the domain the advecting velocity will be small and spatial differencing errors will dominate time differencing errors. This provides the motivation for semi-Lagrangian methods. Though implicit time differencing has been used to represent part of the advection process in spectral models, it is difficult to do this in finite difference methods where a nonlinear equation would have to be solved at each timestep. Furthermore, implicit advection schemes slow down transport processes in regions where speeds are high enough for explicit methods to be unstable.

If three time levels of data are available, the leapfrog method is natural. If only two time levels are available, then single step advection schemes cannot be used with centred spatial differencing as they are unstable. Two-step schemes must be used, the most common being the Heun (or second order Runge-Kutta) scheme. This takes the form

$$\theta^* = \theta(t) - \Delta t(\mathbf{U}.\nabla\theta(t))$$

$$\theta(t+\Delta t)=\theta(t) - \tfrac{1}{2}\Delta t(\mathbf{U}.\nabla(\theta(t)+\theta^*)).$$

Though this scheme is still weakly unstable, the growth rate is proportional to $(U\Delta t/\Delta x)^4$. This is found to be negligible in practice. The use of shape preserving spatial differencing schemes avoids the need for a two-step time differencing scheme.

5.2 Spatial approximation

Conservative spatial differencing schemes for advection on a regular grid are built up as illustrated below for equations (16) and (17) on the C grid.

$$\frac{\partial\theta}{\partial t} = u\overline{\delta_x\theta}^{x} + v\overline{\delta_y\theta}^{y} + \omega\overline{\delta_p\theta}^{p} \tag{22}$$

$$0 = \delta_x u + \delta_y v + \delta_p \omega \tag{23}$$

Adding gives

$$\frac{\partial\theta}{\partial t} = \delta_x(u\overline{\theta}^{x}) + \delta_y(v\overline{\theta}^{y}) + \delta_p(\omega\overline{\theta}^{p}). \tag{24}$$

Equation (24) is in conservation form, so that the total amount of θ is conserved under advection. Multiplying (22) by θ and adding (23) gives

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\theta^2) = \delta_x(u\tilde{\theta\theta}^{x}) + \delta_y(v\tilde{\theta\theta}^{y}) + \delta_p(\omega\tilde{\theta\theta}^{p}).$$

where $\tilde{\theta\theta}^{x} = \theta(x-\frac{1}{2}\delta x)\theta(x+\frac{1}{2}\delta x)$. This shows that $\int\theta^2$ is conserved under advection. Equation (24) can also be interpreted as an approximation to the integral of equation (17) reound the grid box, with the rate of change of θ being made up of a sum of fluxes. Equation (22) can also be interpreted as changing θ by a sum of 'advective fluxes', though the use of the word 'flux' in this context is no longer strictly correct. A wider class of conservative advection schemes can now be constructed by redistributing these 'advective fluxes'. These schemes will still conserve $\int\theta$ but not $\int\theta^2$. Thus a fourth order accurate approximation to the first term on the right hand side of (22) is

$$\tfrac{7}{6}(U\overline{\delta_x\theta}^{x}) - \tfrac{1}{6}(U\overline{\delta_x\theta}^{3x}).$$

It is possible to construct arbitrarily high order schemes this way.

The spectral method can be considered as the limit of higher and higher order centred finite difference methods. It is also possible to construct shape preserving schemes. Both these topics are treated in the papers by Smolarkiewicz in this volume.

6. SMOOTHING AND FILTERING

6.1 Grids on the sphere

Almost all finite difference atmospheric models use the latitude longitude grid on the sphere. The east-west grid-length reduces as the poles are approached, and the curvature of the spherical polar coordinate system becomes severe. Attempts to alleviate this be reducing the east-west resolution as the poles were approached gave poor quality results. The standard solution is therefore to retain the full grid, but to remove the small scales by filtering so that the effective resolution does not increase towards the poles and the timestep can be maintained at a value appropriate for the resolution elsewhere. A minimum resolution to be retained is chosen. The filtering can then be done by Fourier filtering, where the data are transformed into Fourier components, either truncated at a wavenumber determined by the chosen resolution or the amplitudes scaled down above this wavenumber, and transformed back to gridpoints. Methods of implementing this were compared by Takacs et al. (1981). Alternatively, linear filters, Shapiro (1975), can be used to achieve the same effect as scaling the amplitudes of Fourier components.

Another recent approach discussed by Purser (1988) is to retain all the grid points but to use a 'skipped grid' to compute east-west differences, thus avoiding the timestep restriction. When shape-preserving advection schemes are used, filtering cannot be used as the properties of the advection scheme are nullified. These schemes are normally implemented as a sequence of one-dimensional schemes, and the east-west step has to be repeated several times with a shorter timestep at the lines of latitude close to the poles.

6.2 Subgrid models

The subgrid model included in equations (1) to (8) is partly a smoothing operator and is an essential part of the equations. In gridpoint models, it also has to do the job of eliminating scales where the finite

differencing errors are large. For instance, centred advection schemes give large errors for wavelengths less than four gridlengths. The inability to enforce boundedness constraints on the advected data can lead to solutions which become unstable, even if the timestep is within the limit predicted by linear theory. The use of schemes which conserve the square of advected quantities, or conserve energy or enstrophy (vorticity squared) reduces this risk. However, though it may be possible to run a model which uses these schemes stably without a subgrid model, the results will not be realistic because the subgrid model is an essential part of the equations. The use of shape preserving advection schemes implies that the bounds on the data are preserved. Such schemes always contain some smoothing as well, in particular they will reduce the integrated variance of the data when some of this variance occurs below the scales that can be resolved by the grid. This allows them to do the job of the subgrid model, and a separate model is no longer needed.

The smoothing schemes used are usually either linear filters or approximations to analytic diffusion terms. The filters aredesigned to eliminate the smallest scales completely and low or high order filters can be used according to how rapidly the effect is desired to diminish as the scale increases, Shapiro (1975). Lower order filters are usually required as the resolution of models increases. Diffusion terms can also be of various orders. The simplest form is ∇^{2n} applied to each variable. Note that for n>1, such a term is not guaranteed to reduce the total variance. However, n=2 is the most popular for large scale models. as resolution is increased it is necessary to choose n=1 to prevent spurious increases of variance. Nonlinear forms can also be used. In order to retain conservation, they should be written in the form $\nabla.K\nabla(\nabla^{2n})$ where K is a data dependent coefficient. Many attempts have been made to derive K from turbulence theory, but in practice simple schemes seem to work as well as more elaborate ones in large scale models.

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