SHORT-TERM DYNAMICS OF FORECAST ERROR COVARIANCES

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Summary: Elementary statistical theory tells us that successful data assimilation requires some knowledge of the covariance structure of forecast errors. Optimal interpolation (OI) methods are based upon quasistationary, homogeneous, geostrophic approximation of the forecast error covariance. Like OI, three-dimensional variational methods require the forecast error covariance at each analysis time. Four-dimensional variational methods require it at the beginning of each assimilation interval. Kalman filter techniques attempt to describe and utilize the detailed evolution of the forecast error covariance throughout the assimilation cycle.

Accurate description of the forecast error covariance is difficult because, for nonlinear dynamics in three space dimensions, the dynamical equation for covariance evolution involves six space dimensions, to express the coordinates of each pair of points being correlated. A theory is now being developed that attempts to return the dynamics of covariance evolution to three dimensions, by considering only the dynamics of globally varying but locally defined fields, such as variances and correlation lengths. We describe basic results obtained so far, and we discuss the direction of work that remains to be done. While the theory is still in an early stage of development, a few of its implications are immediate, and we focus some attention on them.

1. INTRODUCTION

The potential importance of evolving second-moment forecast error statistics (covariances), along with the forecast itself, has long been recognized. Epstein (1969) and colleagues, for example, in their work on stochastic-dynamic prediction, showed that evolving covariances would allow one to calculate the additional terms that convert ordinary prediction equations into equations for the ensemble mean. In contemporary numerical weather prediction (NWP) practice, we use a model to calculate a single realization of the atmospheric state, which is what we call a forecast. In stochastic-dynamic prediction, in addition one evolves covariances according to their dynamics, and feeds this information back into the model, in such a way that the model calculates the ensemble mean state rather than a single realization. Such dynamical ensemble mean forecasting is what Monte Carlo forecasting methods (Leith, 1974; Hoffman and Kalnay, 1983) attempt to approximate.

Covariance evolution is also important in data assimilation. In conventional optimal interpolation (OI: Bergman, 1979; Lorenc, 1981), the forecast error covariance field is explicitly required at each analysis time. Rather than actually evolving this field, however, it is approximated as a quasistationary field. The covariance is also required in 3D and 4D variational approaches to data assimilation (Lewis and Derber, 1985; Le Dimet and Talagrand, 1986; Courtier and Talagrand, 1987), which are the main subjects of this Workshop. The problem of "cycling" 4D-VAR (see the paper by Courtier and Thépaut in the present volume) is in fact primarily the problem of evolving the forecast error covariance. In estimation-theoretic approaches to data assimilation, covariance evolution actually plays a central role. The dynamical equation for covariance evolution, in fact, is one of the five so-called Kalman filter equations (e. g., Jazwinski, 1970).

It is known that forecast error covariances behave in a manner considerably different from that specified in OI schemes (Cohn and Parrish, 1991; Daley, 1992a). It is also known that this difference can considerably degrade the performance of OI (Cohn et al., 1981; Dee, 1991; Todling, 1992). Still, evolving covariance fields according to their governing dynamical equations, especially in an operational environment, remains an open research problem.

There are two primary, immediate difficulties. First of all, the covariance evolution equation requires specification, or estimation, of the model error covariance. Rather little work has been undertaken in this area so far; see the papers of Dee et al. (1985) and Daley (1992b), and the paper of Dee in the present volume. Second, solving the covariance equation is exceedingly expensive, at least if done by brute force. The reason for this is that if we consider a forecast model having N degrees of freedom ($N \sim 10^6$ for current NWP models), then the corresponding covariance evolution model will have about $N^2/2$ degrees of freedom. This is equivalent to carrying out N/2 forecasts, which is clearly not a good idea.

There are other difficulties. For example, if the first two problems can be successfully overcome, then it will become necessary (and possible) to consider moment closures of higher than second order. The choice of moment closure influences the duration for which covariance dynamics are accurate. Second-order closure is widely held to be adequate for short-term forecasts, but is probably not valid for the extended range (Epstein, 1969; Lacarra and Talagrand, 1988; Rabier and Courtier, 1992).

The computational expense of covariance evolution was addressed by Parrish and Cohn (1985), who proposed and tested a discrete banded approximation, in which correlations past a certain prespecified small distance were set to zero at each time step. This approach was later abandoned in the work of Cohn and Parrish (1991), where it was discovered that correlations can in fact be significant over considerable distances.

Dee (1991) suggested simplifying the covariance equation by basing it on simplified dynamics, such as mass advection. This still involves computational effort of $O(N^2)$, although with a very much smaller constant in front of the N^2 than for complete dynamics.

Another approach is simply to carry out covariance evolution at lower resolution. For this purpose a spectral formulation is probably best, and this was tried by Le Moyne and Alvarez (1991). In this approach, however, one cannot account for (and hence cannot correct) errors in the small spatial scales present in the forecast model but not in the covariance model. A simple remedy would be to approximate the missing small-scale part of the spectral forecast error covariance matrix by a time-independent, diagonal matrix, as is done for the entire spectrum in the SSI system at NMC (Parrish and Derber, 1992), while evolving the large-scale part. This has not been tried yet, but would be a rather simple and direct route to useful approximate covariance evolution. It could be implemented easily in any spectrally-based 3D or 4D variational assimilation system.

A recent paper (Cohn, 1992; hereinafter referred to as C92) has initiated another approach to approximate covariance evolution. The basic idea is to derive and solve time-dependent partial differential equations for the aspects of covariance fields in which we are most interested, such as variances and correlation length scales. Here we want to summarize this new approach and its basic results, and to discuss limitations of the approach and future directions.

2. LOCAL COVARIANCE EVOLUTION

2.1 Assumptions

Essential to the approach is to deal directly with the covariance evolution equation as a partial differential equation (PDE). If the underlying model dynamics are governed by a PDE system in three space dimensions, then covariance evolution is governed by a PDE system in six space dimensions, since covariances are functions of pairs of points, each specified by three coordinates.

To get started, C92 takes the basic dynamics to be univariate, i.e., governed by a single, scalar PDE. This case is very much simpler than the multivariate case, and extension of the theory to multivariate dynamics, discussed later on, is not trivial. Further, C92 treats mostly first-order quasilinear dynamics (e.g., Courant and Hilbert, 1962). That is, the governing dynamics can be arbitrarily nonlinear in the dependent variable (and in the independent variables, of course), but linear in the differential operators $\partial/\partial t, \partial/\partial x, \ldots$. Thus, dispersive processes are not considered, for example, but nonlinear and linear advection equations are.

For ease of notation, let us consider here dynamics in just one space dimension. None of the results are particularly influenced by dimensionality; see Sec. 2.5 for further discussion and see

C92 for results in two and three dimensions. Then the covariance equation involves two space dimensions. Under standard statistical assumptions, including second-order moment closure, it is shown in C92 that this covariance equation has the form

$$\frac{\partial P}{\partial t} + a(x_1,t)\frac{\partial P}{\partial x_1} + a(x_2,t)\frac{\partial P}{\partial x_2} + \left[b(x_1,t) + b(x_2,t)\right]P = Q \ , \tag{1}$$

where $P = P(x_1, x_2, t)$ is the forecast error covariance, $Q = Q(x_1, x_2, t)$ is the model error covariance, and x_1 and x_2 are the two points being correlated. The coefficients a and b are actually functions of the mean state $\overline{w}$,

$$a(x,t) = f(\overline{w}\,(x,t)\,, x, t)\,, \tag{2}$$

$$b(x,t) = g(\overline{w}\,(x,t)\,, x, t)\,, \tag{3}$$

and coupling with the dynamical equation for the mean state (not shown here) makes the coupled system nonlinear.

2.2 Local covariance dynamics

It is our common experience that forecast error covariance functions usually exhibit most of their structure where the two points in question are separated by relatively small distances; at large distances, covariance functions usually fall off to zero. Yet covariance evolution is governed by essentially hyperbolic equations on a closed (six-dimensional) space, so that propagation would ordinarily be expected to fill up all of the space with significantly nonzero values. One surmises, therefore, that covariance equations must have a special form that tends to keep covariance structure local, i.e., limited to pairs of points that are fairly close together. Further, one ought to be able to derive equations for this local structure.

For eq. (1), it turns out that what is special is simply that the coefficients of $\partial / \partial x_1$ and $\partial / \partial x_2$ are one and the same function, $a(\cdot\,, t)$, evaluated at x_1 and x_2, respectively. The theory of characteristics is used in C92 to show why this keeps covariance structure local: where $|\,x_1 - x_2\,|$ is small, the direction of propagation is nearly parallel to the plane $x_1 = x_2$.

This suggests rewriting (1) in the rotated coordinate system defined by

$$x = \frac{1}{2}(x_1 + x_2)\ , \tag{4}$$

$$\xi = \frac{1}{2}(x_1 - x_2). \tag{5}$$

In this system, the direction of propagation is (nearly) the direction of the x-axis, at least for $|\xi|$ (i..e., $|x_1 - x_2|$) small. This will enable us to get most of the information we want from the 2D PDE (1) by solving instead only 1D PDEs in the variable x, i.e., the PDEs for the local structure that we are after. Defining now

$$\tilde{P}(x, \xi, t) \equiv P(x_1(x, \xi), x_2(x, \xi), t) = P(x + \xi, x - \xi, t) \ , \tag{6}$$

and defining $\tilde{Q}$ similarly, eq. (1) transforms as

$$\frac{\partial \tilde{P}}{\partial t} + \frac{1}{2}\left[a(x+\xi, t) + a(x-\xi, t)\right]\frac{\partial \tilde{P}}{\partial x}$$

$$+ \frac{1}{2}\left[a(x+\xi, t) - a(x-\xi, t)\right]\frac{\partial \tilde{P}}{\partial \xi} + \left[b(x+\xi, t) + b(x-\xi, t)\right]\tilde{P} = \tilde{Q} \ . \tag{7}$$

This equation is already "almost" one-dimensional, since the coefficient of $\partial \tilde{P}/\partial \xi$ here vanishes for $\xi = 0$ and is therefore small when $|\xi|$ is small, assuming that a(x,t) is a continuous function of x.

To obtain the local equations, expand $\tilde{P}$ in a Taylor series about $\xi = 0$:

$$\tilde{P}(x, \xi, t) = P_o(x, t) + \frac{1}{2}\xi^2 P_2(x, t) + \frac{1}{4!}\xi^4 P_4(x, t) + \ldots \ , \tag{8}$$

where

$$P_j \equiv \left. \frac{\partial^j \tilde{P}(x, \xi, t)}{\partial \xi^j} \right|_{\xi = 0} \ , \tag{9}$$

and where we have assumed that $\tilde{P}$ has at least four continuous derivatives at $\xi = 0$, in which case the first and third derivatives must vanish by the symmetry property of covariance functions. Assume a similar expansion for $\tilde{Q}$. Observe that

$$P_o(x, t) \equiv \tilde{P}(x, 0, t) = P(x, x, t) \tag{10}$$

is simply the forecast error variance, while $P_2(x,t)$ expresses the curvature of the covariance function at $\xi = 0$. Substituting the expansions for $\tilde{P}$ and $\tilde{Q}$ into (7) and equating terms independent of ξ gives the forecast error variance equation

$$\frac{\partial P_o}{\partial t} + a\frac{\partial P_o}{\partial x} + 2bP_o = Q_o \ , \tag{11}$$

where $a = a(x,t)$ and $b = b(x,t)$. Here $Q_o = Q_o(x,t)$ is the model error variance. Equating terms proportional to ξ^2 gives the second-order equation

$$\frac{\partial P_2}{\partial t} + a\frac{\partial P_2}{\partial x} + 2(a_x + b)P_2 + a_{xx}P_{ox} + 2b_{xx}P_o = Q_2 \ , \tag{12}$$

while equating terms proportional to ξ^4 gives the fourth-order equation

$$\frac{\partial P_4}{\partial t} + a\frac{\partial P_4}{\partial x} + 2(2a_x + b)P_4 + 6a_{xx}\frac{\partial P_2}{\partial x} + 4(a_{xxx} + 3b_{xx})P_2 \tag{13}$$
$$+ a_{xxxx}P_{ox} + 2b_{xxxx}P_o = Q_4 \ .$$

One can, of course, continue this process.

Each of these equations is a one-dimensional equation (three-dimensional in the "real" case), to be solved along with the one-dimensional (three-dimensional) equation for the mean state. The doubled dimensionality of the original covariance equation (1) has been removed. Each equation is coupled to the previous equations, but not to any of the following ones. Thus the set of local equations can be terminated at any order, without approximation, depending in principle on the amount of accuracy with which an approximation to the local covariance structure is sought.

2.3 An example

As a very simple example, consider the nonlinear advection equation

$$u_t + uu_x = 0 \ . \tag{14}$$

In C92 it is shown that the mean equation is then

$$\bar{u}_t + \bar{u}\,\bar{u}_x + \sigma\,\sigma_x = 0 \ , \tag{15}$$

where σ^2 is the variance, denoted by P_o earlier. The variance equation (11), written in terms of the standard deviation σ, turns out to be

$$\sigma_t + \bar{u}\,\sigma_x + \bar{u}_x\,\sigma = 0 \ . \tag{16}$$

The higher-order equations like (12) and (13) can also be written, but here we just want to discuss the nonlinear feedback between (15) and (16). The term $\sigma\sigma_x$ is the nonlinear feedback term in (15).

Adding (15) and (16) gives

$$(\bar{u}+\sigma)_t+(\bar{u}+\sigma)(\bar{u}+\sigma)_x=0 \ , \tag{17}$$

while subtracting them gives

$$(\bar{u}-\sigma)_t+(\bar{u}-\sigma)(\bar{u}-\sigma)_x=0 \ . \tag{18}$$

In this simple example we therefore have decoupled nonlinear advection equations for the error bars $\bar{u} \pm \sigma$, from which the mean $\bar{u}$ and standard deviation σ can then be determined. Since the solution of a nonlinear advection equation is always bounded from above by its initial maximum and from below by its initial minimum, it follows readily from (17) and (18) that the mean state $\bar{u}$ and standard deviation σ are individually bounded. This is the nonlinear effect of saturation: initial errors can grow, but they must saturate eventually (e.g., Lorenz, 1982; Dalcher and Kalnay, 1987).

Notice, however, that saturation does not occur if we neglect the nonlinear feedback term $\sigma\sigma_x$ in (15), i.e., if we use the original dynamical equation (14) coupled with (16) for individual realizations u instead of the ensemble mean $\bar{u}$, viz.,

$$\sigma_t+u\sigma_x+u_x\sigma=0 \ . \tag{19}$$

It is well known that every nonconstant solution of (14) must develop a discontinuity, at which $u_x \to -\infty$, assuming a periodic domain for instance, in finite time. The standard deviation σ must then become unbounded at the discontinuity, as is seen by writing (19) in the form

$$\frac{1}{\sigma}\frac{D\sigma}{Dt}=-u_x \ , \tag{20}$$

where $D/Dt \equiv \partial/\partial t + u\partial/\partial x$.

This argument shows that the nonlinear feedback term is essential to account for saturation, at least in our example. Without it, the variance equation displays incorrect behavior, unbounded behavior in this case.

It is important to point out here that the so-called extended Kalman filter (EKF), which is the simplest nonlinear extension of the linear Kalman filter, is in fact based on the use of realizations (14) rather than on the ensemble mean equation (15); cf. Jazwinski (1970). We therefore expect the EKF to be of limited use in NWP, at least for assimilation periods on the order of the error saturation time or longer. Indeed, Evensen (1992) found unbounded variance growth in an EKF experiment, and this may have been due to the mechanism we have just described. The EKF also failed for Miller et al. (1992), who attempted to correct it by accounting for higher moments in the covariance equation, while still using the realization equation instead of the ensemble mean equation to get the state estimate.

If Kalman-like filters are to be used at all for meteorological data assimilation, instead of the EKF it will probably be best to use second-moment (or higher) closure filters, in which an ensemble mean equation is used, and is truncated at the same order as the covariance equation. If one is evolving covariances at all, then the terms to close the mean equation at second order are automatically available anyway, as we have seen. In fact, in the univariate case, the only covariance information that enters the mean equation is the variance field; correlations are not relevant. We have seen that in the univariate case there is in fact an evolution equation for the variance alone, so that this information is easy to come by.

In the 4D-VAR context, these arguments may also eventually come into play. Individual variational assimilation periods are currently envisioned to be fairly short, about two days, which is shorter than the error saturation time at most spatial scales (cf. Dalcher and Kalnay, 1987). Thus the use of the actual model dynamics (a realization) over the assimilation period, as opposed to ensemble mean dynamics, is probably justified. On the other hand, in the context of <u>cycling</u> 4D-VAR, i.e., of applying it over long periods of time, it may well be necessary to find a way to account for the difference between realization dynamics and ensemble mean dynamics.

2.4 <u>Correlation dynamics</u>

Since there is a variance equation (11) for univariate dynamics, it is straightforward to use it with the covariance equation (1) to obtain a dynamical equation for the <u>forecast error correlation</u> field $C(x_1, x_2, t)$, defined by

$$C(x_1, x_2, t) \equiv \left[P_o(x_1, t)\, P_o(x_2, t)\right]^{-1/2} P(x_1, x_2, t) \ . \tag{21}$$

The correlation evolution equation is

$$\frac{\partial C}{\partial t} + a(x_1, t)\frac{\partial C}{\partial x_1} + a(x_2, t)\frac{\partial C}{\partial x_2} + \frac{1}{2}\left[\frac{Q_o(x_1, t)}{P_o(x_1, t)} + \frac{Q_o(x_2, t)}{P_o(x_2, t)}\right] C$$
$$= \left[\frac{Q_o(x_1, t)\, Q_o(x_2, t)}{P_o(x_1, t)\, P_o(x_2, t)}\right]^{1/2} T(x_1, x_2, t) \ , \tag{22}$$

where T is the model error correlation. This equation, equations derived from it, and versions in more space dimensions, were studied in detail in C92. Here we discuss some of the main results.

One can draw some conclusions immediately from the form of (22). For example, the fact that the coefficient of the undifferentiated term in (22) is positive (recall that Q_o and P_o are variances) implies that the response to the initial condition is transient, as in the case of the ordinary differential equation $\dot{y} + ay = f$ with $a > 0$. The influence of the initial (previous analysis) error correlation decays while the effect of the model error correlation grows. At sufficiently long times, the model error correlation dominates the forecast error correlation.

This is a hopeful result, at least to the extent that a similar statement might also hold for complicated, multivariate dynamics. Over infrequently observed regions at least, it may be more important to know the model error correlation than to evolve the previous analysis error correlation. While techniques to estimate model error statistics are just now beginning to be explored, as mentioned in the Introduction, it will probably prove less costly to estimate model error correlations reasonably well than to evolve detailed features of analysis error correlations. The latter display sharp gradients due to the act of observing, while the former are not affected by observations. Model error correlations are affected by nonlinear internal model dynamics, though, and therefore cannot be expected to be stationary or homogeneous (see the paper by Dee in this volume). A substantial part of the model error may be stationary and homogeneous, on the other hand, and estimable by relatively inexpensive methods (Daley, 1992b).

Since the correlation equation (22) has the same form as the covariance equation (1), a set of equations for local correlation dynamics like (11) - (13) can be obtained from it by the same coordinate rotation and series expansion. The zero-order equation says simply that

$$C_o(x, t) = T_o(x, t) = 1 \ , \tag{23}$$

i.e, that correlations equal unity at zero separation, as we know. The second-order equation is

$$\frac{\partial C_2}{\partial t} + a\frac{\partial C_2}{\partial x} + \left(2a_x + \frac{Q_o}{P_o}\right)C_2 = \frac{Q_o}{P_o}\left[T_2 - \left(\frac{\partial}{\partial x}\log\frac{Q_o}{P_o}\right)^2\right] . \tag{24}$$

$C_2 = C_2(x, t)$ is the second derivative of the forecast error correlation field at zero separation, and so it should be expected that $C_2(x, t) < 0$. In fact, all solutions of this equation can be shown to satisfy this constraint, provided $C_2(x, 0) < 0$ and $T_2(x, t) < 0$, T_2 being the second derivative of the model error correlation field at zero separation.

The <u>correlation length field</u> $L = L(x, t)$ is defined by

$$L^2 \equiv \frac{1}{-C_2} . \tag{25}$$

The correlation length in OI schemes, for example, has generally been taken to be a constant scalar parameter, in fact the primary parameter describing a correlation function (e.g., Daley, 1991, Sec. 4.3, 5.2, 5.3). By using (25) in (24) one obtains a (nonlinear) dynamical equation for the actual forecast error correlation length field $L(x, t)$. Examination of that equation shows that $L(x, t)$ tends to relax, along characteristics, toward a critical correlation length field $L_* = L_*(x, t)$, defined by

$$L_*^2 \equiv \frac{1 + 2a_x P_o / Q_o}{-T_2 + \left(\dfrac{\partial}{\partial x} \log \dfrac{Q_o}{P_o}\right)^2} \, . \tag{26}$$

This result shows how an important feature of forecast error correlations can be expected to behave. It also suggests a potential avenue for approximating forecast error correlation fields, simpler still than the direct use of (24), which is already simpler than the complete correlation equation (22). Depending on the relaxation time scale, an approximation in which L is simply replaced by L_*, rather than evolved according to (24), may prove worthwhile.

2.5 Generalizations and limitations

The extension of the theory described here to two and three space dimensions is straightforward and is treated in C92. There is still a variance equation, for example. The number of higher-order equations increases with dimensionality. For example, in n dimensions there are n(n+1)/2 second-order equations, in place of the single equation (12) or (24).

Thus, for univariate dynamics in three dimensions, one would need eight equations to evolve the mean, variance and correlation length fields. For the practical purposes of data assimilation, Daley (1992a) has suggested that from the complete covariance field it may suffice to evolve accurately only the variance field. His arguments are strengthened by ours, especially if approximations to the correlation length field such as that discussed following (26) turn out to be workable. To evolve just univariate mean and variance fields in any number of dimensions requires only two equations.

The presence of diffusive (even-order) or dispersive (odd-order higher than first) derivative terms in the original dynamics does not fundamentally alter our theory, although an approximation does become necessary. Such higher-order derivative terms produce corresponding derivative terms in the covariance equation. The coordinate rotation and series expansion still work, but the resulting dynamical equations become forward-coupled. In the presence of second or third derivative terms, for example, the variance equation becomes coupled to the second-order equation. Thus an approximation to the coupling term becomes necessary in order to terminate the system. To terminate at the variance equation, a simple approximation would be to take the correlation length to be fixed, or given by some other approximation such as $L = L_*$; cf. (26). The successfulness of such approximations remains to be tested.

Multivariate dynamics are almost always dispersive, and will require approximation not unlike that just described. For general multivariate dynamics one can still obtain local covariance equations in half the number of dimensions of the complete covariance equation (unpublished work), as we have done for univariate dynamics. The zero-order equation still gives variances, and is coupled to a first-order equation. Current work focuses on approximations here, such as a quasigeostrophic one, that might soon make variance evolution, and perhaps evolution of some higher-order fields also, a practical reality.

3. CONCLUSIONS

We have outlined the beginnings of a theory directed toward meaningful simplification of the dynamical equation for forecast error covariance evolution. The theory reduces this six-dimensional equation to three-dimensional equations that describe the space-time dynamics of fundamental features of the forecast error covariance, such as forecast error variances and correlation lengths. The theory helps us to understand the basic properties of forecast error covariance propagation. It is also aimed toward practical application in data assimilation, either for variational or Kalman-like approaches, as well as for stochastic-dynamic prediction. For such applications to be fruitful, much testing needs to be carried out, beginning with very simple models. Also much theoretical work remains to be accomplished, especially in the multivariate case.

Two general conclusions about our approach itself can be drawn here. Our approach, while based on simple second-moment closure (third-moment discard), is fully nonlinear and produces from the original governing dynamics an ensemble mean equation coupled nonlinearly to either a complete covariance equation or to our reduced, local system of equations deduced from the covariance equation. The conventional extended Kalman filter (EKF), on the other hand, is derived from linear theory and as a result uses the original governing dynamics instead of ensemble mean dynamics, with no nonlinear feedback from the covariance equation. Thus it uses an inconsistent closure, namely second-moment discard for the mean equation and third-moment discard for the covariance equation. We saw for the nonlinear advection equation that this inconsistent closure is not able to account for the nonlinear effect of error saturation, and in fact leads (incorrectly) to unbounded growth of variance, while consistent, second-moment closure does account for saturation.

We conclude that the EKF is probably of limited use in meteorological data assimilation, since it fails to account for error saturation, which is a zero-order effect of nonlinearity. Instead, internally consistent, second-moment (or higher) closure filters will be more useful. These remarks also apply to 4D variational approaches to data assimilation, in which the actual governing dynamics rather than ensemble mean dynamics are applied over assimilation intervals. While these intervals are planned to be shorter than characteristic saturation time scales, if 4D-VAR is cycled by patching together these intervals in some way, and if the actual governing dynamics rather than ensemble mean dynamics continue to be used in each interval, then some approximate means of accounting for error saturation will become necessary.

The results we have outlined in this paper depend fundamentally on the fact that we have posed the problem of covariance dynamics directly as a problem in partial differential equations (PDE) rather than in ordinary differential equations (ODE) or even in discrete dynamics (difference equations). The results depend crucially, for example, on a change of independent variables. The understanding of why this change of variables works depends on the theory of characteristics. Neither characteristics nor the change of independent variables exists in the ODE or discrete dynamics case. Indeed, while our results must still hold approximately for semi-discretizations (ODE) or full discretizations (difference equations), they are certainly not obvious from an ODE or difference equation framework, and they probably cannot be derived in such a framework without fairly explicit reference to the underlying PDE. Nearly all work in the meteorological data assimilation arena to date, be it OI, variational, or estimation-theoretic, has been posed discretely or at least semi-discretely. Some questions concerning data assimilation, such as those of observability, can actually be proven to require a discrete framework for their answers (Cohn and Dee, 1988).

The results of the present paper allow us to conclude, though, that to enable further theoretical progress in data assimilation, and especially to develop more effective computational algorithms, it will be important to continue exploring PDE formulations of the data assimilation problem. Without working directly on the continuum, important features of our problem are hidden and efficient algorithms may remain undiscovered. PDE formulations of the data assimilation problem will allow us to employ all the classical techniques of studying and simplifying complicated continuum dynamics, including scaling arguments, physical reasoning, and various mathematical devices.

ACKNOWLEDGMENTS

I wish to thank R. Daley for pointing out to me the paper by Evensen, and its connection with the example of Sec. 2.3. The work described in this paper has benefitted tremendously from discussions at various times with R. Daley, D. Dee, J. Derber, E. Kalnay, R. Ménard, D. Parrish, and R. Todling. The manuscript was typed by Q. Philpot. I thank P. Courtier, T. Hollingsworth and E. Kooij-Connally for their gracious hospitality during the Workshop.

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