ON THE DEVELOPMENT OF A CELL-INTEGRATED SEMI-LAGRANGIAN SHALLOW WATER MODEL ON THE SPHERE

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1.   INTRODUCTION

In the so-called cell-integrated semi-Lagrangian model considered here trajectories are constructed backward in time from the corner points of a grid cell, thereby defining the extent of the cell at a previous time. The predictive variables being advected are considered to be averaged values over the cell rather than point values. The determination of the averaged values over the irregular cells at the previous time level is called a remapping. By choosing integral invariants (e.g. mass, angular momentum, total energy) as the predicted variables the application of the cell-integrated scheme gives a model which conserves exactly these invariants in a locally consistent way.

A full set of baroclinic model equations, based on the primitive ones, for such a model was set up in Machenhauer (1994). With an explicit time scheme this system would conserve exactly mass, total energy, angular momentum, entrophy, humidity and last but not least important any passive tracer (e.g. chemical constituents) included in the model. In a subsequent paper Machenhauer and Olk (1996) investigated the numerical stability of simple one-dimensional shallow water model versions for which they also designed and tested versions with semi-implicit time extrapolation.

After the design of the baroclinic cell-integrated semi-Lagrangian system it was realized that a somewhat similar system had been developed for the Navier-Stokes equations by Hirt et al. (1974) and had been applied to hydrodynamical problems. One of the differences is that we will use a semi-implicit time extrapolation whereas Hirt et al.(1974) introduced an iterative procedure to adjust pressure gradient forces. More similar to our cell-integrated scheme are the advection schemes introduced and tested by Rančić (1992) and Laprise and Plante (1995). Although we did not know their work when our scheme was initially developed, the cell-integrated model system we propose may be considered as an extension of their schemes to the complete system of meteorological equations.

In the present paper we design a cell-integrated semi-Lagrangian model for the 2D shallow water equations on the globe. The main purpose of the paper is the presentation of a new efficient remapping procedure which will be tested in the shallow water model.

An explicit version of the model is presented in section 2. In section 3 we present the new remapping procedure and finally in section 4 we conclude the paper and give some specifications of the semi-implicit system which will be used in the shallow water model tests planned.

2.   EXPLICIT MODEL EQUATIONS

In the cell-integrated semi-Lagrangian method we use as far as possible the meteorological equations on their basic Lagrangian forms which express the fundamental conservation laws, i.e. for the shallow water equations conservation of mass, total energy and angular momentum.

For frictionless flow on a flat earth, they become

$$\frac{d}{dt}(h\delta A) = 0 \tag{1}$$

$$\frac{d}{dt}(E\delta A) = -\delta A \nabla \cdot (\frac{1}{2}gh^2 \vec{V}) \tag{2}$$

$$\frac{d}{dt}(M\delta A) = -\delta A \frac{\partial}{\partial \lambda}(\frac{1}{2}gh^2) \tag{3}$$

with $E = \frac{1}{2}(V^2 + gh)h$ and $M = (au\cos\varphi + \Omega a^2 \cos^2\varphi)h$. λ and φ are longitude and latitude, respectively, the constants g, Ω and a are gravity, the angular velocity and the radius of the earth, respectively. h is the height of the fluid, which apart from a constant factor, the density, is equal to the mass in a column of unit cross section area. δA is the cross section area of a fluid column or a cell we follow and thus $h\delta A$ is proportional to the mass of the cell.

The first equation, the continuity equation, is saying simply that the mass in the cell is constant with time. As usual u and v are the eastward and northward components of the horizontal velocity vector (which is constant with height in the cell). $V = \sqrt{u^2 + v^2}$ is the wind speed. $E\delta A$ and $M\delta A$ are the total energy, kinetic plus potential, and the angular momentum of the cell we follow, respectively. The energy equation, the second one, expresses that the total energy of the cell can change only as a result of its work performed against the pressure forces from the surrounding fluid and finally the third equation expresses that the angular momentum is changing only due to pressure gradient torques.

We will use a so-called reduced latitude-longitude grid as indicated in Fig. 1 where for each grid cell we predict averaged values of h, E and M over the cell. As illustrated in Fig. 2 we use a three-time-level scheme with time step Δt and determine the departure points at $t-\Delta t$ of trajectories which end up at the corner points of the grid cells. The trajectories are assumed to be great circles and are determined using the winds at time level t and the usual iterative procedure.

In the case of explicit time stepping we get the following discrete form of the equations (1)-(3):

$$h_{i,j}^+ \Delta A_j = \overline{h_{i,j}^-} \delta A_{i,j}^- \tag{4}$$

$$E_{i,j}^+ \Delta A_j = \overline{E_{i,j}^-} \delta A_{i,j}^- + g\Delta t \left[\sum_{k=1}^{4} h_k^2 (\vec{V} \cdot \vec{n})_k \delta l_k \right]_{i,j}^0 \tag{5}$$

$$M_{i,j}^+ \Delta A_j = \overline{M_{i,j}^-} \delta A_{i,j}^- + g\Delta t \left[\sum_{k=1}^{4} sign(\vec{i} \cdot \vec{n}) h_k^2 \delta\mu_k \right]_{i,j}^0 , \tag{6}$$
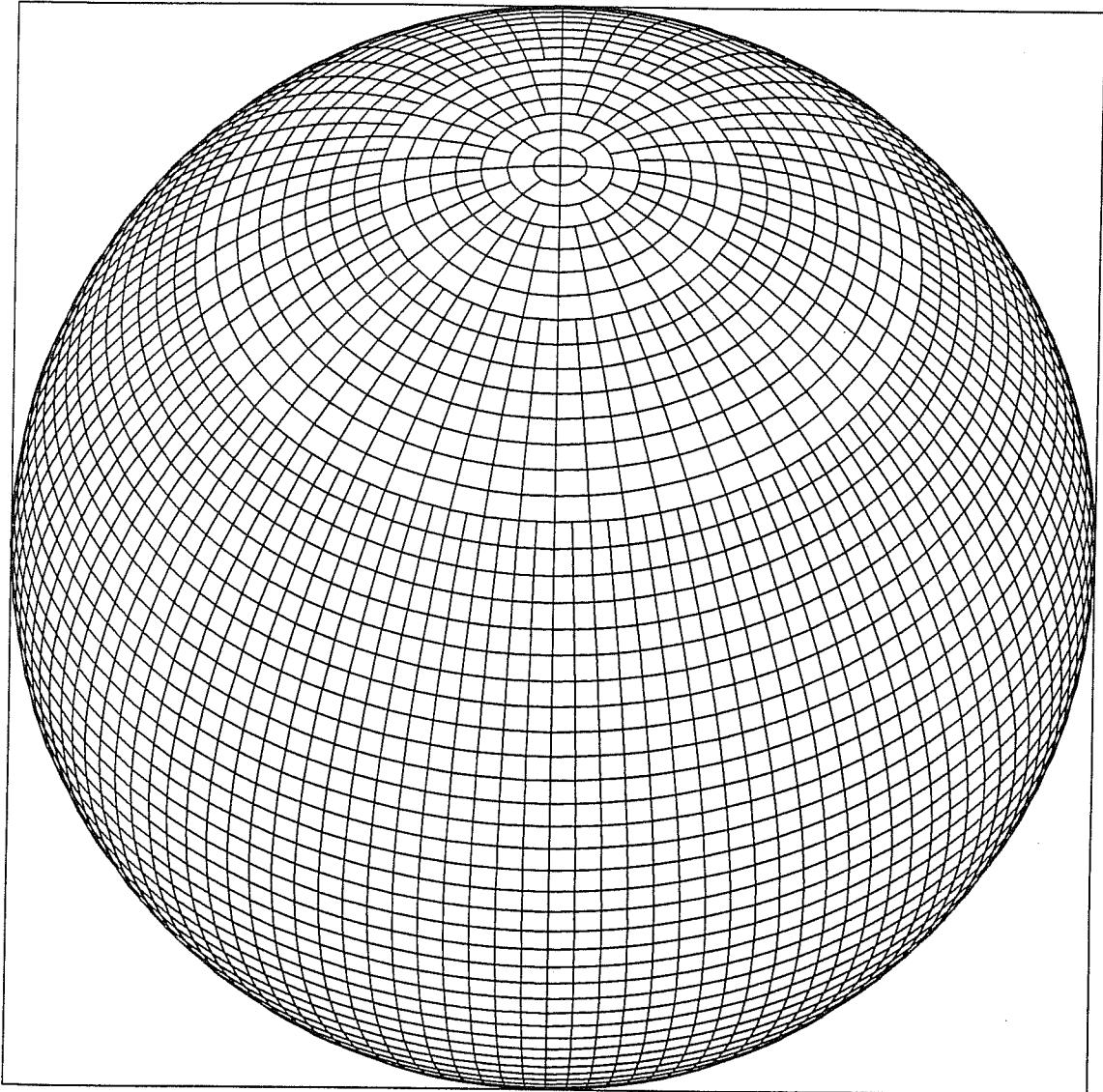
Fig. 1 The reduced grid. In the present example there are 128 grid cells in longitude at the Equator and 64 grid cell rows in latitude. (After Rasch, 1993).
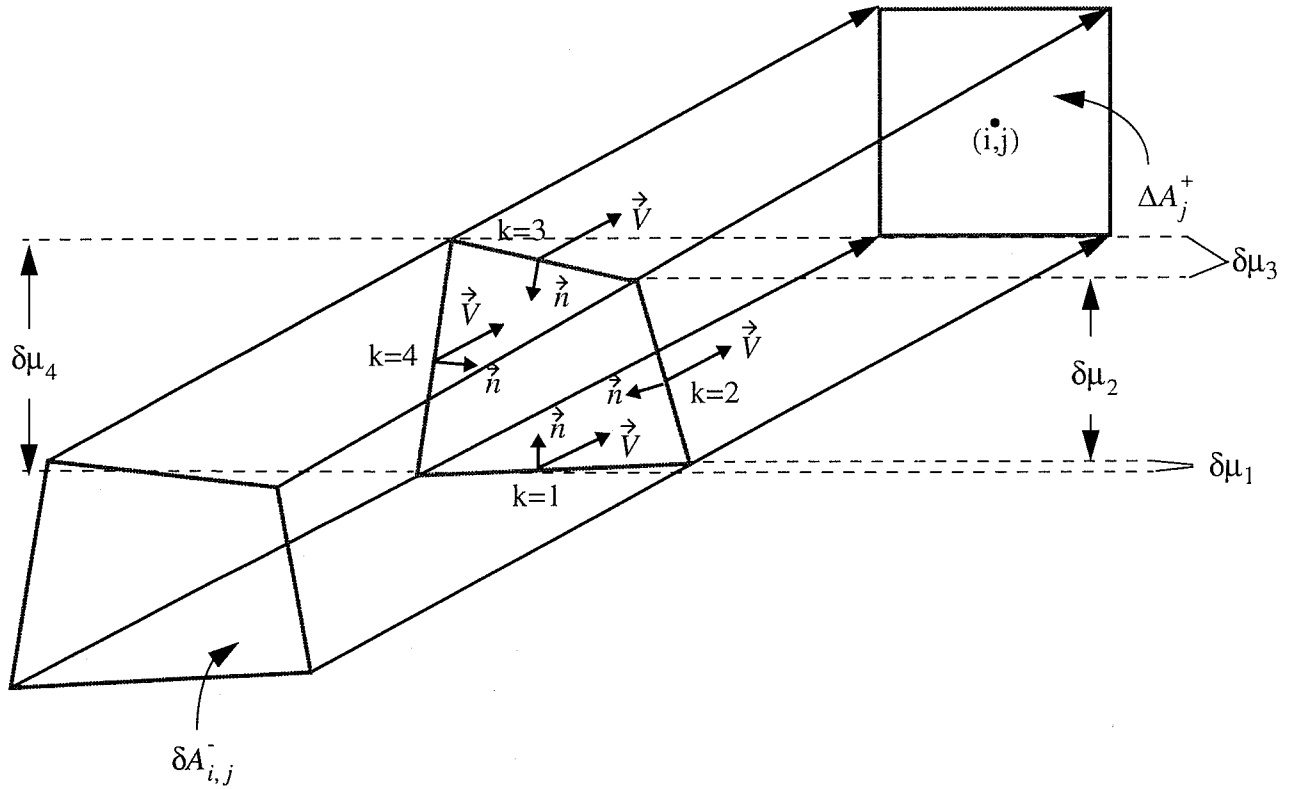
Fig.2   Illustration of quantities used in equations (4)-(6).

where we define

$$E^+_{i,j} = \frac{1}{2}\left(\left(u^+_{i,j}\right)^2 + \left(v^+_{i,j}\right)^2 + gh^+_{i,j}\right)h^+_{i,j} \tag{7}$$

and

$$M^+_{i,j} = au^+_{i,j}\cos\varphi_{i,j} + \Omega a^2\cos^2\varphi_{i,j}. \tag{8}$$

Referring to Fig. 2, i and j denote the grid cell number in the zonal and meridional direction, respectively. The superscripts "+", "0" and "−" denote the time levels t+Δt, t and t−Δt, respectively.

The time stepping proceeds as follows: At first $h^+_{i,j}$ is determined from (4), then $u^+_{i,j}$ is calculated from $M^+_{i,j}$ determined by (6) using (8). Finally, $v^+_{i,j}$ can be derived from $E^+_{i,j}$ determined by (5) using (7). $h^+_{i,j}$, $u^+_{i,j}$ and $v^+_{i,j}$ are considered to be averaged values over the cell area. When needed for interpolations to specific points they are, however, assumed to represent point values located in the centre of the cell.

The second term on the right hand sides of (5) and (6) approximates the right hand sides of (2) and (3), respectively. Each of the terms in the sums are evaluated at the mid points of a cell side at its position at time level t. To evaluate these terms the velocity $\vec{V}$ must be interpolated to the grid points of the four sides and the length δl_k and the inward normal vectors $\vec{n}_k$ at the midpoints must be determined. The calculation of h^2 and $\vec{V}$ is done as usual using bicubic interpolation in the latitude-longitude grid. In order to determine δl_k and $\vec{n}_k$ we need to define the curves that constitute the sides of the cell at time level t. We chose to define the sides as straight lines between the corner departure points in a λ-μ coordinate system, where μ = sinφ.

In the first term on the right hand sides of (4)-(6) a bar denotes an averaged value over the cell area. In the continuity equation for instance $\bar{h}^-_{i,j}\delta A^-_{i,j}$ denotes the integral of h at time level t−Δt over the (i,j)-cell area, $\delta A^-_{i,j}$ (see Fig. 2):

$$\overline{h^-_{i,j}\delta A^-_{i,j}} = a^2 \iint\limits_{\delta A^-_{i,j}} h^-\cos\varphi d\varphi d\lambda = a^2 \iint\limits_{\delta A^-_{i,j}} h^- d\lambda d\mu . \tag{9}$$

We use the λ-μ coordinate system for the evaluation of such integrals because of the simple cartesian-coordinate-like expression we obtain for the integrals and because as a consequence of the choice above the sides in the quadrangle of area $\delta A^-_{i,j}$ are straight lines in this coordinate system.

The integrals which shall be performed in order to evaluate the first right hand side terms in (5) and (6) are similar to (9) except that in instead of h⁻, E⁻ and M⁻ must be integrated over the cell area $\delta A^-_{i,j}$. The procedure to be used to compute the integrals in all three cases is the same. We shall describe it only for the mass integral considered in (9).

3.   AN EFFICIENT REMAPPING PROCEDURE

At time level t−Δt we have given the area mean values $\bar{h}^-_{i,j}$ in each of the "regular" grid cells (see Fig. 3). In order to evaluate the integral in (9) over grid cells in the irregular grid we have to define analyt-
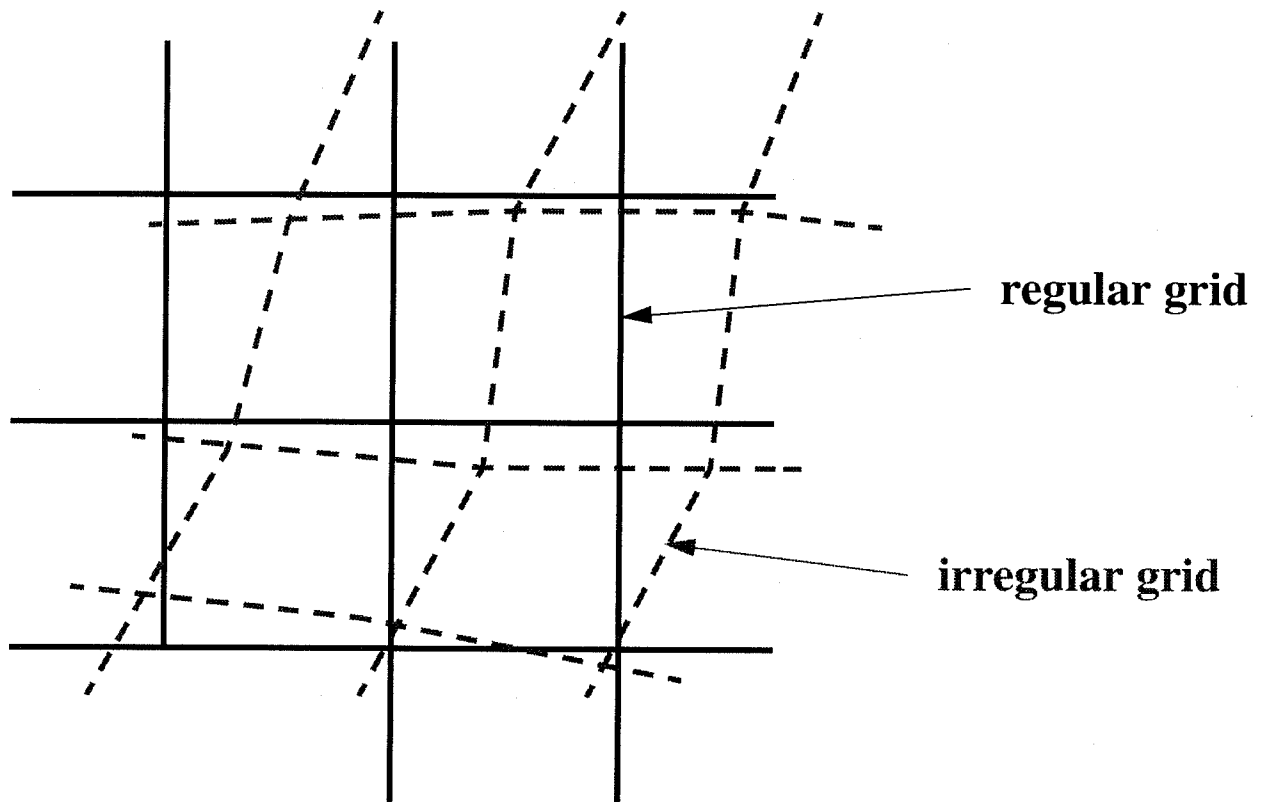
Fig.3 Illustration showing irregular grid cells at t–Δt which at t+Δt end up in regular grid cells.

ical two-dimensional functions $\psi_{i,j}(\lambda,\mu)$ in each regular cell. Each function must have the correct mean value i.e. it must satisfy

$$\bar{h}_{i,j} = \frac{1}{\Delta A_j} \iint_{\Delta A_j} \psi_{i,j}(\lambda, \mu) \, d\lambda d\mu \ . \tag{10}$$

The simplest possibility is to assume piecewise constants for $\psi_{i,j}$, that is, $\psi_{i,j}(\lambda, \mu) = \bar{h}_{i,j}$. This would imply discontinuities at the cell boundaries and rather severe damping during the integration, especially of small scales. To reduce the damping the assumption of a higher order polynomial than the piecewise constant one is necessary. Piecewise biparabolic functions were considered by Rančić (1992) to be a good compromise between the desired properties and the computational cost to determine them in each cell. Such functions can be written as

$$\psi_{i,j}(x, y) = a_2(y) x^2 + a_1(y) x + a_0(y) \tag{11}$$

where

$$a_2(y) = a_{22}y^2 + a_{21}y + a_{20}$$

$$a_1(y) = a_{12}y^2 + a_{11}y + a_{10}$$

$$a_0(y) = a_{02}y^2 + a_{01}y + a_{00}$$

x and y being auxiliary coordinates

$$x = \frac{\lambda - \lambda_{i-1/2,j-1/2}}{\Delta\lambda_j} \quad \text{and} \quad y = \frac{\mu - \mu_{i-1/2,j-1/2}}{\Delta\mu_j} \tag{12}$$

which vary between 0 and 1 within the cell from a starting point in the southwest corner of a cell. Such functions were used by Rančić (1992) in his tests of a cell-integrated semi-Lagrangian advection scheme. The nine a-coefficients were determined in such a way that (10) is satisfied and so that interpolated values at the cell corner points and side mid points were fitted exactly. The determination of the coefficients for all the cells followed by the so-called remapping, i.e. the determination of the integrals in (9), turns out to be very time consuming. In his comparison with a usual semi-Lagrangian cubic interpolation scheme Rančić (1992) found for pure advection that the cell integrated scheme with the biparabolic functions (11) gave slightly better results but needed 2.5 times more computer time.

From the start of the present project it was therefore obvious that a more efficient method should be found, as far as possible without sacrifying the good performance with biparabolic representations. The alternative we propose in the present paper seems to be such a method although it remains to be tested how much faster it is in practice.

By remapping we mean the determination of the mean value by means of the integral (9), for each cell of the irregular grid at t−Δt, given the mean values of all cells in the regular grid. In the new remapping procedure we compute at first in each of the cell corner points values of what we will call the accu-

mulated mass $H(\lambda,\mu)$, which is the total mass in the area south and west of a point as indicated for point A in Fig. 4. (For a point on the southern hemisphere H is the total mass north and west of that point. The procedure we describe here for the northern hemisphere becomes symmetric with respect to the Equator.) H is easily computed in the regular grid points (the corner points of the regular cells) since the mean value of h in the cells of the regular grid is known. We notice that from the accumulated mass H in the four corner points of an arbitrary rectangle ABCD (see Fig. 4) the mass m in that rectangle can be computed simply by

$$m = H(C) - H(B) - H(D) + H(A) .$$ (13)

This is not of particular interest if the corner points of the rectangle are some regular grid points. It would then be just as easy to add up the known mass in each cell within the rectangle. (13) can, however, be applied to any arbitrary rectangle, where the corner points are not necessary grid points. This will be used in the new remapping procedure. In order to do that, we must be able to compute H in any point. We shall see how that can be done.

Consider now Fig. 5. In order to compute H in point C we first have to determine H(B). This is done by adding the mass m1 in the rectangle $A_\mu ABB_\mu$ to H(A),

$$H(B) = H(A) + m1$$ (14)

where H(A) is the known accumulated mass at the cell corner point A and m1 is the sum of contributions from a number of cells. Each of these contributions can be written as

$$(\Delta m1)_{i,j} = \int_0^{y1}\int_0^{} \psi_{i,j}(x, y)\, dx\, dy$$

$$= \int_0^y (\frac{1}{3}a_2(y) + \frac{1}{2}a_1(y) + a_0)\, dy$$

$$= \int_0^y (b'_2 y^2 + b'_1 y + b'_0)\, dy$$ (15)

where by using (9) we find that

$$b'_2 = \frac{a_{22}}{3} + \frac{a_{12}}{2} + a_{02}$$

$$b'_1 = \frac{a_{21}}{3} + \frac{a_{11}}{2} + a_{01}$$

$$b'_0 = \frac{a_{20}}{3} + \frac{a_{10}}{2} + a_{00} .$$

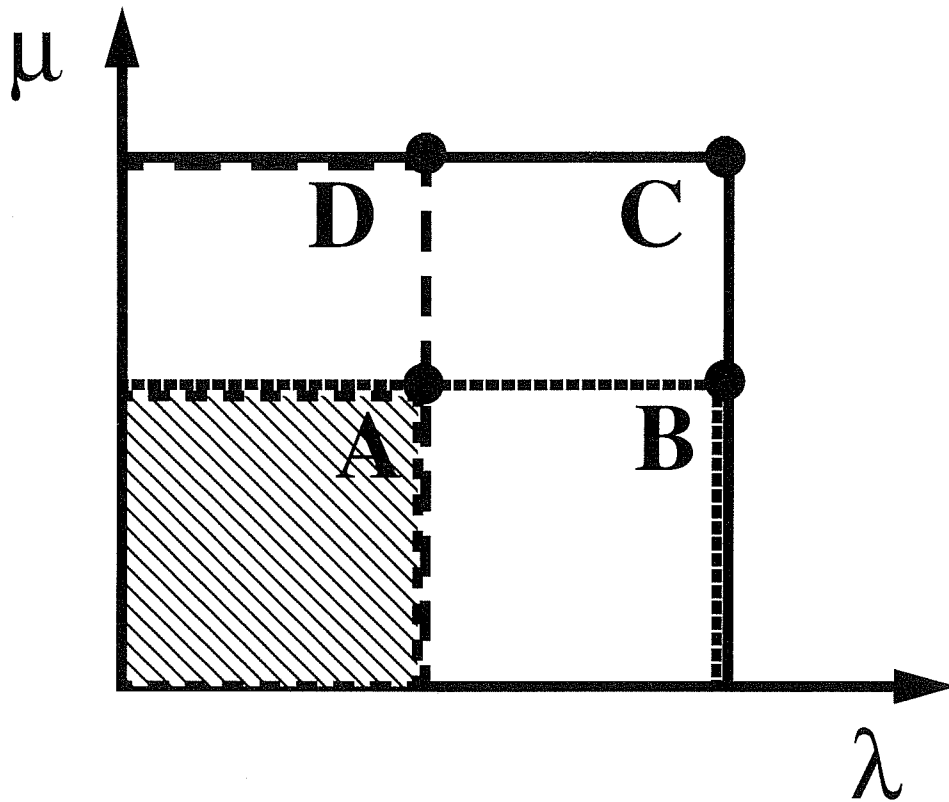After adding up all the contributions $(\Delta m1)_{i,j}$ in the row we obtain

Fig. 4 The accumulated mass for point A, H(A), is the mass in the hatched area. The accumulated mass in the corner points of the rectangle ABCD determines its mass (equation (13)).
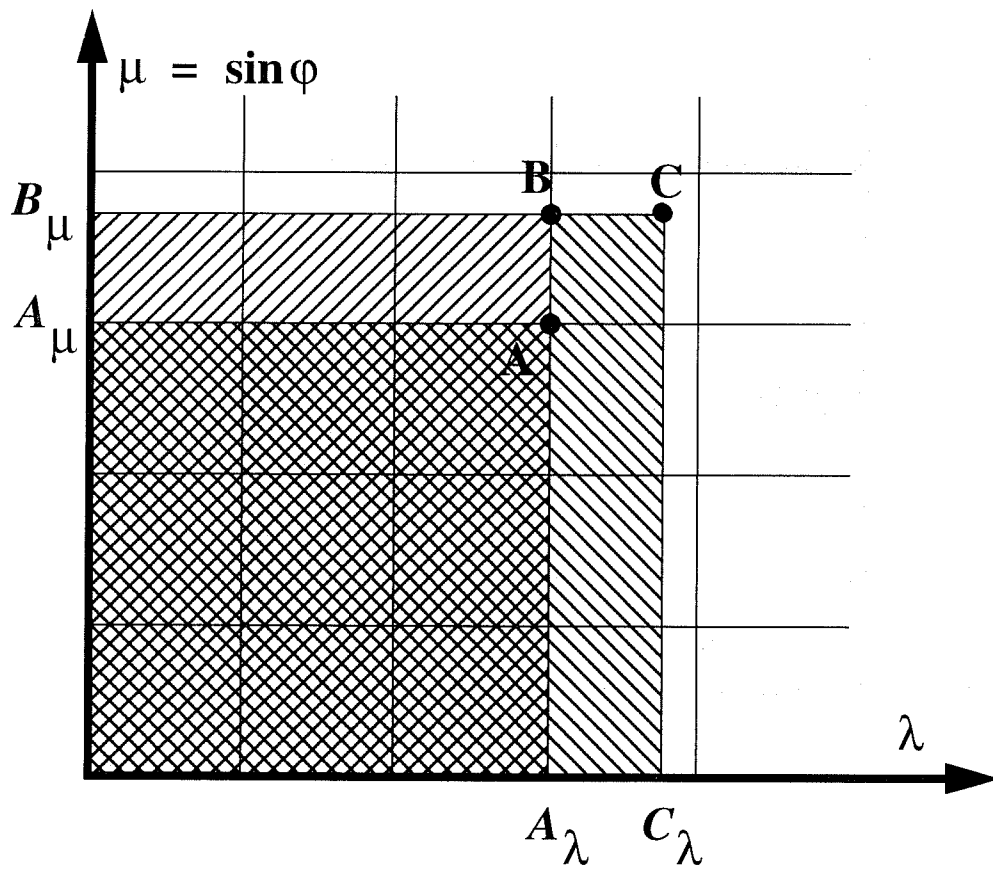
Fig. 5 Illustration to the procedure used in the determination of the accumulated mass H(C) in an arbitrary point C.

$$m1 = \int_0^y (b_2 y^2 + b_1 y + b_0)\, dy \tag{16}$$

where the coefficients b_i (i=0,1,2) are determined by the b'_i's and so by all the coefficients a_{ij} (i,j=0,1,2) of the contributing cells in the row. It would be computational expensive to use (16) with b_i's determined from all the a_{ij}'s, because, as mentioned above, it would be expensive to fit biparabolic functions (11) in every cell of the regular grid. Instead of this we propose the following less expensive method.

We do, indeed, use (16) to compute H(B), but we derive the coefficients b_i from zonally accumulated values to fit the centre value and be continuous at the cell boundaries (y=0 and y=1). With this procedure we obtain the same values as we would have got by a determination from the a_{ij}'s.

The next step is to calculate H(C) by adding the mass m2 in the rectangle $A_\lambda C_\lambda CB$ to H(B). Like for m1 it consists of contributions $\Delta m2$, in this case from a column. Using a derivation similar to (15) we get

$$H(C) = H(B) + m2$$

and

$$m2 = \int_0^x (c_2(y)\, x^2 + c_1(y)\, x + c_0(y))\, dx$$

where the coefficients c_i (i=0,1,2) are determined directly from meridionally accumulated values, again fitting a central value and being continuous at cell boundaries.

Thus for the determination of the accumulated mass in an arbitrary point, (C in our example) we need to fit only two one-dimensional parabolic functions even though the result is the same as if we had used biparabolic functions in all the cells.

We will see now how this economical procedure can be used in the design of an efficient remapping algorithm. Consider the lefthand quadrangle ABCD in Fig. 6. We want to determine the mass m in this quadrangle by using the fact that it is equal to the mass in the outer rectangle, who's corner points are A, BA, BD and AD, minus the mass in the four hatched triangles and minus the mass in the small inner rectangle with corner points C, BC, BD and CD. Using the result above, that the mass in an arbitrary rectangle ABCD is given by (13), and assuming that the mass in each hatched triangle is equal to half the mass of the rectangle it is a part of, we find the following expression for the mass m in the quadrangle (with the notation of Fig. 6):

$$m = H(A) + H(BD) - H(AD) - H(BA)$$
$$-\frac{1}{2}(H(CD) + H(DC) - H(D) - H(C)$$
$$+ H(D) + H(A) - H(AD) - H(DA)$$
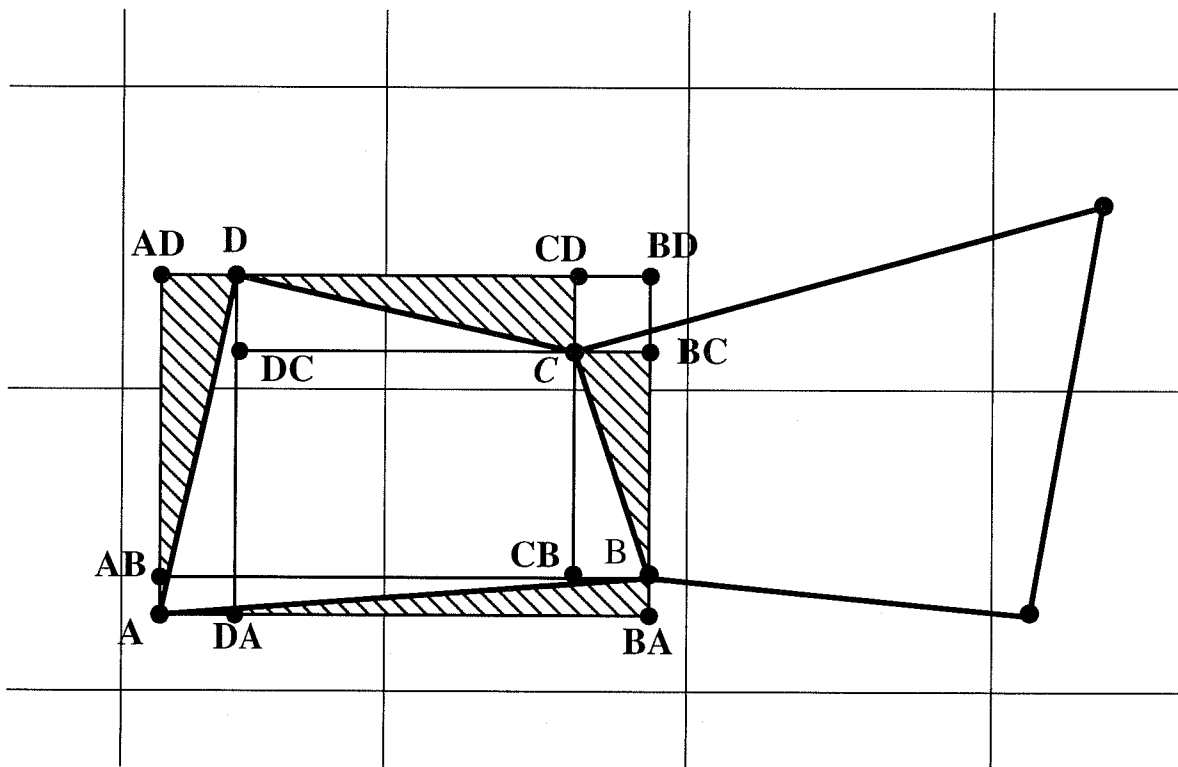$$+ H(B) + H(A) - H(AB) - H(BA)$$
$$+ H(BC) + H(CB) - H(C) - H(B))$$

Fig. 6 Illustration to the derivation of equation (17) giving the mass in an arbitrary quadrangle ABCD.

$$-(H(BD) + H(C) - H(CD) - H(BC))$$
$$= \tfrac{1}{2}(H(AB) - H(BA) + H(BC) - H(CB)$$
$$+ H(CD) + H(DC) - H(DA) - H(AD)) . \tag{17}$$

Thus, from the final expression it is seen that the mass in the quadrangle is simply a sum of four contributions, one from each side. The contribution of one side is given by the difference between the accumulated mass H in the two corner points at the ends of the diagonal in the rectangle in which the side in question is the other diagonal. The mass of the right neighbour quadrangle in Fig. 6 is also the sum of four terms, one from each side. The contribution from the western side of the quadrangle (i+1,j) is of the same magnitude but opposite sign as the contribution from the eastern side of the quadrangle (i,j).

There are special cases in which the determination of the cell mass deviates from that given by (17). One such special case is illustrated in Fig. 7. Here a departure cell is shown, which ends up at t+Δt in one of the triangular cells near the pole. Using the same procedure as above the following expression is derived for the mass

$$m = 0.5 (H(AB) - H(BA) + H(BC) - H(CB) + H(CA) - H(AC)) . \tag{18}$$

Again, the mass is simply a sum of contributions from each of the sides and each contribution is of the same form as those in (17).

There are other special cases, namely when the departure cells are intersected by one or both of the axes, $\lambda = 0$ or $\mu = 0$. In such cases the mass is determined separately for each quadrant and then added to give the total cell-mass. In all cases the contributions from the cell sides are of the same form as in (17) and (18). Apart from such special cases the remapping of each quadrangle cell requires the determination of eight accumulated mass values H, but because each contribution from a side is used also by a neighbour cell only four are needed per cell. Thus, the total number of H calculations needed for each remapping is approximately four times the number of cells.

Approximately two one dimensional parabolic functions must be fitted for each H calculation, that means approximately eight such fits per grid cell, which probably involves the same number of arithmetic operations as a bicubic interpolation. This indicates that with the above described remapping procedure the cell-integrated scheme may become as efficient as the traditional semi-Lagrangian methods with the additional advantage that integral constraints are fulfilled exactly due to a locally consistent transport of mass and other invariants.

4. CONCLUDING REMARKS

A new efficient remapping procedure was set up in the context of a cell-integrated semi-Lagrangian shallow water model on the sphere. For simplicity an explicit (leap-frog) time extrapolation was chosen. To be competitive, however, a time scheme which allows longer time steps must be used. In
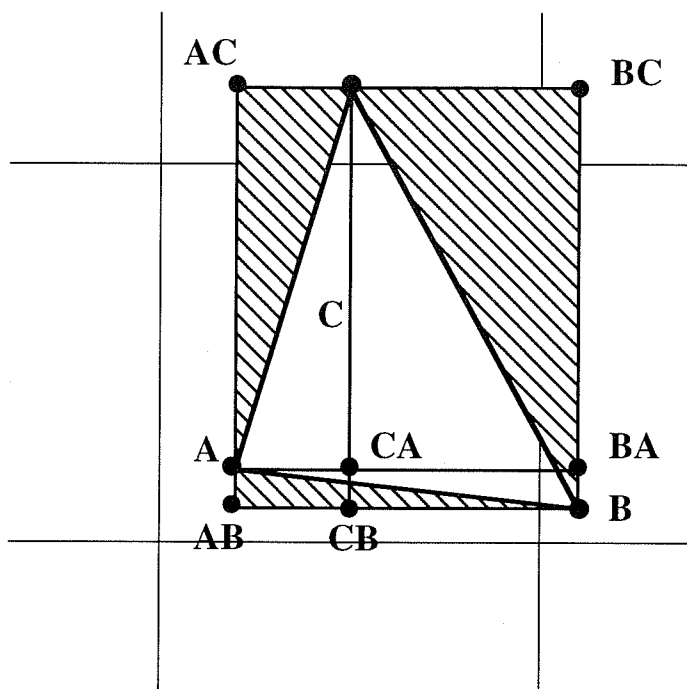
Fig. 7 Illustration to the derivation of equation (18) giving the mass in an arbitrary triangle.

Machenhauer and Olk (1996) it was shown for a one-dimensional shallow water system, that a semi-implicit scheme can be designed for a cell-integrated model when momentum as well as when total energy is chosen as the exactly conserved quantity. (In a one-dimensional model only two quantities, mass and an additional one, can be conserved exactly.) In the 2D shallow water system three quantities can be conserved and the obvious choice is mass, angular momentum and total energy as we have done for the explicit system described in section 2. It seems, however, that it seems not possible to chose total energy as one of the predictive variables when we want to use the semi-implicit scheme. This is due to the fact that we cannot linearize the expression for kinetic energy in two dimensions as we could do it in one dimension. For a semi-implicit model corresponding to the one described in section 2 we consequently have to choose an alternative third equation instead of (3). In the model we are setting up we have decided to use the v-component of the momentum equation in vector form instead of (3). The form of the momentum equation we will use is

$$\frac{d}{dt}((Uh\vec{i} + vh\vec{j})\,\delta A) = -\delta A \nabla \frac{1}{2}gh^2 \tag{19}$$

where $U = u + 2a\Omega\cos\varphi$. $\vec{i}$ and $\vec{j}$ are unit vectors pointing towards east and north, respectively. It has been derived from the vector form of the momentum equation proposed by Rochas (1990) and Temperton (1994) in which the Coriolis term is included into the advective part. We will use only the v-component which is derived from the discrete version of (19) (taking into account, of course, the changes of the orientation of the local coordinate system as one moves along the trajectories (Ritchie, 1988)).

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