TECHNICAL MEMORANDUM

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A spurious mode in the 'Lorenz' arrangement of f and T which does not exist in the 'Charney Phillips' arrangement

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February 1995

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Preface

This note demonstrates the existence of a spurious mode in the 'Lorenz' arrangement of ϕ and T that is commonly used in the vertical differencing of primitive equation models. The note also shows that the 'Charney-Phillips' arrangement of ϕ and T does not have a spurious mode.

Written in spring 1975, the note was filed in a dusty drawer because the planning staff of ECMWF had more pressing concerns. The text presented here is essentially as it was originally written. No attempt has been made to take account of more recent work by A Arakawa and his associates at UCLA, and by M Cullen and his associates at UKMO. It is being circulated now because the results in Fig 4 may be relevant to current concerns in stratospheric modelling and data assimilation.

1. INTRODUCTION: AN EXAMPLE OF SPURIOUS GEOSTROPHIC BALANCE

The thermal wind relation in pressure coordinates is

$$f\frac{\partial u}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial y}$$
$$f\frac{\partial v}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial x}.$$

It follows that, if a geostrophic wind is independent of pressure, then the temperature field is independent of position on a constant pressure surface and vice-versa. A non-zero geostrophic wind which is independent of pressure can only be balanced by a surface pressure field. This latter result does not obtain in certain finite difference formulations of the primitive equations. In some linearised finite-difference formulations of a σ-coordinate model one can specify a geostrophic wind which is independent of height, and an arbitrary surface pressure field; one can then find a thermal field such that the fluid is in a steady state.

Consider a σ-coordinate model with N levels at which velocities U, V, geopotential ϕ and temperature T are carried. Let $T = \mathrm{T}(p) + T'$ and $\phi = \Phi(p) + \phi'$. Then the linearised pressure force E_k at each level k is given by

$$E_k = \nabla\phi_k' + R\mathrm{T}_k\nabla\ln p_* \qquad 1.1$$

where p_* is the surface pressure. The hydrostatic relationship is usually written in the form

$$\phi_k' - \phi_{k+1}' = \alpha_k(T_k' + T_{k+1}') \qquad 1.2$$

where k is 1 at the top level and N at the bottom level. For example, in the scheme used by *Corby, Gilchrist and Newson* (1972)

$$\alpha_k = \frac{R}{2}\ln\frac{\sigma_{k+1}}{\sigma_k}$$

Eq 1.2 can be written in the form

$$\underline{\phi}' = g\,\underline{T}' \qquad 1.4$$

where now $\underline{\phi}'$, $\underline{T}'$ are column vectors $[\phi_k]$, $[T_k']$ and g is a square matrix. g is specified by

$$g = \begin{bmatrix} \alpha_1, & \alpha_1+\alpha_2, & \alpha_2+\alpha_3, & ..\alpha_{N-1}+\alpha_N \\ & \alpha_2, & \alpha_2+\alpha_3, & ..\alpha_{N-1}+\alpha_N \\ & & \alpha_3 & . \\ & & . & . \\ & & . & \alpha_{N-1}+\alpha_N \\ 0 & & & \alpha_N \end{bmatrix} \qquad 1.5$$

Suppose we consider the equations 1.1 and 1.2 as a problem to determine the T_k', knowing F_k. This problem is under-determined as we must specify $\ln p_*$ before a unique solution can be found. Once p_* is specified we can always find a solution. A fortiori, in the case of a geostrophic wind which is independent of height, so that F_k is independent of height, we can specify $\ln p_*$ arbitrarily and find a thermal field which will give a steady state solution.

To consider the nature of this solution we consider the simple case of $F_k = 0$, i.e. zero pressure gradient at each model level. Let us assume an unbounded plane geometry, and expand the column vectors $\underline{\phi}'$, $\underline{T}'$ as Fourier Series

$$\underline{\phi}' = \sum \hat{\underline{\phi}}\, e^{i(kx+ly)}$$
$$\underline{T}' = \sum \hat{\underline{T}}\, e^{i(kx+ly)}$$

where $\hat{\underline{\phi}}$, $\hat{\underline{T}}$ are of course column vectors.

Then 1.1 and 1.2 imply

$$0 = \hat{\underline{\phi}} + \mathrm{T}(R\ln\hat{p}_*)$$
$$\hat{\underline{\phi}} = g\hat{\underline{T}}$$

where T is $[\mathrm{T}_k]$, the mean temperature field. The dependence of $\hat{\underline{T}}$ on $\ln\hat{p}_*$ is given by

$$\hat{\underline{T}} = -g^{-1}\,\mathrm{T}\,(\ln\hat{p}_*)R. \qquad 1.6$$

It is easy to verify that, when g is of the quite general form (1.5), then g^{-1} is given by

$$g^{-1} = \begin{bmatrix} \frac{1}{\alpha_1}, & -\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right), & \left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right) & & \cdot \\ & \frac{1}{\alpha_2}, & -\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right) & & \cdot \\ & & \frac{1}{\alpha_3}, & -\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right) & \\ & & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & \frac{1}{\alpha_{N-1}}, & -\left(\frac{1}{\alpha_N}+\frac{1}{\alpha_{N-1}}\right) & \\ & & \cdot & \frac{1}{\alpha_N} & \end{bmatrix} \qquad 1.7$$

Moreover it is easy to verify that the row-sums of g are all of magnitude $\frac{1}{\alpha_N}$ and alternate in sign. Thus, if N is even, then $N/2$ row-sums have positive sign and $N/2$ row-sums have negative sign. If N is odd, then $[N/2]$ row-sums have negative sign and $[N/2]+1$ rows have positive sign.

In the case when the mean field is isothermal, so that T is independent of height, we see from 1.6 that $\hat{T}$ has the same magnitude in each layer and changes sign from layer to layer. If the surface pressure consisted of a pure sinusoid, then a temperature field of the form shown in Fig 1 would balance the surface pressure field in that the net pressure force at each level would be zero.

Tables 1 and 2 show g, g^{-1} for the case of 10 equally spaced layers for the finite difference scheme specified by 1.3. Table 3 shows T and $g^{-1}T$ for a standard atmosphere in this same model. The two-grid vertical wave nature of the solution is still dominant.

We tested these ideas with a primitive equation model. Since we wish to refer to this model several times, let us call it the 2D model. It is a two-dimensional $\left(\frac{\partial}{\partial y}\equiv 0\right)$ plane-geometry version of the UK Universities Atmospheric Modelling Group's Model (Garp pub 13). We used 10 vertical levels, cyclic boundary conditions in X of period L=10,000 km and no "physics" i.e. no internal smoothing, no convection scheme, no heat sources apart from those to be specified explicitly later. There were 20 points in the X-direction.

For the present experiment we initialised with a northward velocity V which was of the form $V = 50.\left(Sin\frac{2\pi}{L}X\right)$ m/sec. This we balanced by a thermal field according to Eq 1.1 with $lnp_*=0$. We took the Coriolis parameter $f=10^{-4}$. After 5 days of integration the fields had hardly changed. The maximum surface pressure departure from its initial value was ~.002 mb.

In a second experiment we balanced a vertical two-grid wave in the temperature field (of amplitude 10 K) by a surface pressure field (~2 mb), with zero wind initially and $f=0$. After 9 days of integration there was again very little change in the fields. Thus the linearised arguments we presented above carry over to the non-linear model.

These results help to explain the difficulties mentioned by *Hoskins and Simmons* (1975) when they calculated balanced temperatures from winds, in a model of the type under discussion. Unless care is taken with the specification of surface pressure, large two grid vertical waves will be found in the temperature. In effect Hoskins and Simmons adjusted the surface pressure so that there was no two-grid wave in the temperature. As shown by Table 3, this is a reasonable procedure since $\underline{g}^{-1}\underline{T}$ is almost a pure two-grid wave for a standard atmosphere.

2. A SPURIOUS NORMAL MODE

The solutions of

$$0 = \nabla\underline{\phi}_k' + R\,\underline{T}_k\nabla\ln p_* \qquad 1.1$$

$$\phi_k' - \phi_{k+1}' = \alpha_k(T_k'+T_{k+1}') \qquad 1.2$$

which we have discussed are steady-state free mode solutions of this linearised primitive equation model on a plane, an f-plane or a rotating sphere. All of the other variables u, v, $\dot{\sigma}$ etc are identically zero in this free mode.

General circulation models of the type we are discussing predict u, v, T at each of N levels and also have an equation for surface pressure, $3N+1$ equations in all. Thus in a normal mode calculation there will be N Rossby wave and $2N$ gravity wave solutions, together with one other solution which has no analogue in the continuous primitive equations. This spurious solution is the one we have discussed in section 1 above.

In a normal mode calculation with a non resting basic state the frequency of this spurious mode will not be zero and may well be complex. We have some evidence, to be discussed elsewhere, that this is in fact the case.

3. DIRECT FORCING OF THE SPURIOUS MODE

One may enquire if this mode can be excited in a general circulation model. The linearised equations for such a model may be written in the absence of rotation (*Hoskins and Simmons*, 1975)

$$\frac{\partial}{\partial t}\zeta = 0 \qquad \text{a}$$

$$\frac{\partial}{\partial t}\underline{D} = \nabla^2(\underline{\phi}' + R\,\underline{T}\,lnp_*) \qquad \text{b}$$

$$\frac{\partial}{\partial t}\underline{T}' = \underline{Q} - \underline{\tau}\underline{D} \qquad \text{c} \qquad 3.1$$

$$\frac{\partial}{\partial t}\ln p_* = \underline{JD} \qquad \text{d}$$

$$\underline{\phi}' = g\underline{T}' \qquad \text{e}$$

where ζ, D, T', ϕ', Q are column vectors for vorticity, divergence, temperature, geopotential and diabatic heating. p_*, T, g are as before, τ is a square matrix containing the thermodynamics and $\underline{JD}$ is the vertical integral, or weighted sum, of D. Let a subscript 0 denote values at an initial instant t_o. Let $D_o \neq 0$ and suppose that

$$\underline{\phi_o} + R\,\mathrm{T}\,lnp_{*_o} = g\,\underline{T_o'} + R\,\mathrm{T}\,lnp_{*_o} = 0.$$

Then provided the diabatic heating Q is given by

$$\underline{Q} = g^{-1}\,\mathrm{T}(\underline{JD_o}) + \tau\,\underline{D_o}, \qquad 3.2$$

the equations 3.1 have the solution

$$\underline{T}' = \underline{T_o'} + g^{-1}\,\underline{T}(\underline{JD_o})t \qquad 3.3$$

$$(lnp_*) = lnp_{*_o} - (\underline{JD_o})t \qquad 3.4$$

$$\underline{D} = \underline{D_o} \qquad 3.5$$

$$\zeta = \zeta_o. \qquad 3.6$$

In the expression 3.2 for the diabatic heating, the term $\tau\,\underline{D_o}$ is typically small by a factor of 50 or more compared to the first for atmospheric values of D_o. To a good approximation then

$$\underline{T}' = \underline{T_o'} + g^{-1}\,\mathrm{T}(\underline{JD_o})t. \qquad 3.3'$$

$\underline{JD_o}$ is a scalar and $g^{-1}\mathrm{T}$ is almost a pure two-grid wave. Thus if the heating field Q has some roughness, i.e. if Q contains a two-grid wave component, then so long as the vertically integrated divergence is sufficiently large, the two-grid wave in the fluid will grow linearly in time according to 3.3' and the dynamics are short-circuited as shown by 3.5 and 3.6. If the two-grid component is of amplitude Q deg/day, then the vertically integrated divergence must be of the order $Q\alpha_N/\overline{T}$ day^{-1} where $\overline{T}$ is an overall average temperature. For the ten-level model discussed earlier this is of order 2×10^{-9} sec^{-1} for Q=1 deg/day. Moreover $\alpha_N \to 0$ as $N \to 0$, so the spurious mode should be more easily excited as the number of levels is increased.

To test these ideas we performed some experiments with the 2D model in the case of no-rotation. The fluid was initially isothermal at T=275° K and the velocity $U_k = .002\ Sin\frac{2\pi}{L}x$ cm/sec. The initial divergence was then $\sim 2\times10^{-9}\ Sin\frac{2\pi}{L}$ sec^{-1}. We imposed a heating field $Q_k = (-1)^k.\ Cos\frac{2\pi}{L}$ deg/day so that the maximum heating or cooling was 1 deg/day. Fig 2 shows the behaviour with time of the surface pressure, while the inset shows the pressure evolution at the point of largest amplitude. The growth is very nearly linear after the first day and is close to the theoretical rate of .187 mb/day. Table 4 shows the behaviour with time of the temperature in the vertical column where the heating/cooling has largest amplitude. Again the growth is close to the linear theory. Over the same five-day period the maximum velocity in the field increased from 2×10^{-2} cm/sec to something less than 1 cm/sec. If the pressure force were due only to the surface pressure field we would expect to find velocities of order 2 m/sec by this time. These results bear out the validity of the linear theory.

We have repeated this experiment in the situation where the mean U velocity increases from 5 m/sec at the lowest level to 50 m/sec at the top level. The same qualitative effects are found, although the phase lines are tilted by the shear.

We may conclude that in a multi-level model of the type under discussion any roughness, i.e. two-grid component, in the heating field can, in the presence of very weak divergence, lead to the growth of a two-grid wave in the vertical temperature structure.

Depending on the nature of the convection scheme being used in a model of this type, the effect we are discussing could lead to spurious rainfall and convection.

4. ELIMINATION OF THE SPURIOUS MODE

In this section we propose a re-formulation of the finite difference equations which eliminates the spurious mode. We propose that u, v, ϕ be carried at the main levels and that T and $\dot{\sigma}$ be carried at the intermediate levels. Thus T is predicted only at N-1 levels. For didactic purposes the scheme we propose here is as simple as possible. A number of refinements will immediately suggest themselves to the interested reader. Temperature is advected horizontally by the average of the velocities above and below. The original 2D model is very similar to that proposed by *Corby Gilchrist and Newson* (1972). In their formulation the energy conversion term, $\frac{KT\omega}{\sigma}$ is written

$$\left(\frac{\kappa T\omega}{\sigma}\right)_k = -\frac{\kappa T_k}{2}\left[\ln\frac{\sigma_{k+1}}{\sigma_k}\sum_{e=1}^{k}\nabla.p_*\underline{V} + \ln\frac{\sigma_k}{\sigma_{k-1}}\sum_{e=1}^{k-1}\nabla.p_*\underline{v}\right]$$

$$+\frac{\kappa p_*}{a\cos\theta}\left[U_k\overline{\overline{T}_k^{\lambda}\delta_\lambda lnp_*}^{\lambda} + V_k\overline{\cos\theta\overline{T}_k^{\theta}\delta_\theta lnp_*}^{\theta}\right].$$

In the revised 2D model let $T_{k+½}$ denote T at the level between levels k and k+1. Then the expression

$$\left(\frac{\kappa T\omega}{\sigma}\right)_{k+½} = -\kappa T_{k+½}\ln\frac{\sigma_{k+1}}{\sigma_k}\sum_{e=1}^{k}\nabla.p*v$$

$$+\frac{\kappa p_*}{a\cos\theta}\left[U_k\overline{\overline{T}_{k+½}^{\lambda}\delta_\lambda lnp_*}^{\lambda} + V_k\overline{\cos\theta\overline{T}_{k+½}^{\theta}\delta_\theta lnp_*}^{\theta}\right]$$

will lead to energy conservation provided we express

$$(R\ T\ \nabla\ lnp_*)_k = R\ [\overline{\overline{T}_{k+½}^{\lambda}\delta_\lambda lnp_*}^{\lambda}\underline{i} + \overline{\overline{T}_{k+½}^{\theta}\cos\theta\delta_\theta lnp_*}^{\theta}\ \underline{j}].$$

At the lowest layer we use $T_{k-½}$ rather than $T_{k+½}$ in these expressions. The hydrostatic equation is written

$$\varphi_k - \varphi_{k+1} = R\ \ln\frac{\sigma_{k+1}}{\sigma_k}T_{k+½}.$$

To define ϕ_N we assume the lowest layer is isothermal at $T_{N-½}$. At the top and bottom levels

$$\frac{\partial}{\partial\sigma}\dot{\sigma}T = \frac{\dot{\sigma}T}{1.5\Delta\sigma}|_2 \text{ or } |_{N-1}.$$

Several improvements can be made to this scheme. For example, vertical averaging of the winds in the $\frac{\kappa T\omega}{\sigma}$ term and the temperatures in the pressure gradient term would certainly improve the scheme. The simple version suggested here is used to demonstrate that this arrangement of ϕ and T eliminates the spurious mode: if the pressure force is zero at N levels then the surface pressure gradient and the thermal gradient at the N-1 relevant levels must be identically zero.

As a test of the revised scheme and to demonstrate the absence of the spurious mode we have re-run an experiment performed some years ago by E Doron (personal communication). We used the 2D model with the original and the revised vertical schemes. We cooled the lowest layer between 0 and L/2 with a specified function 5 $Sin\left(\frac{2\pi}{L}X\right)$ deg/day. We heated the top layer between $\frac{L}{2}$ and L with the same functional dependence. The situation is illustrated in Fig 3. One would expect a re-distribution of mass within each layer with very little activity in the bulk of the fluid. Figs 4 a and b show the behaviour of the temperature in the two models in the column marked AB in Fig 3.

In the original model it appears (Fig 4a) that the spurious mode is forced to grow over the five-day period. This is borne out by the fact that the temperature at the point of greatest heating is significantly higher than the average temperature for the layer. The disturbance pervades the entire fluid and already by 5 days the integration is beginning to break down.

There is no sign of a spurious mode with the revised model (Fig 4b). The asterisk in the top layer of Figs 4a and b shows the average temperature of the layer at the time in question. The results for the revised model show that there is indeed very little activity in the middle of the fluid. The re-distribution of mass is rapid, particularly in the upper layer where the temperature at the point of most intense heating is very close to the average temperature for the layer. We repeated these experiments with two models each with nine main levels. The results were similar in all essential points. Thus the results do not depend on whether we have an even or odd number of layers.

5. SUMMARY

We have pointed out the existence of a spurious mode in some formulations of the primitive equations. We have discussed the relevance of this mode for violations of the thermal wind relation, for the deduction of temperatures from winds in either geostrophic or balance calculations with these models, and for the study of diabatic effects in these models. We have also suggested a way to eliminate the spurious mode altogether and have shown that this revised scheme works well in a test case.

ACKNOWLEDGEMENTS

I am grateful to A Simmons for helpful comments on the text.

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Corby, G A, A Gilchrist and R L Newson, 1972: A general circulation model of the atmosphere suitable for long period integrations. Quart J Roy Meteor Soc, 98, 809-832.

Hoskins, B J and A J Simmons, 1975: A multi-layer spectral model and the semi-implicit method. Quart J Roy Meteor Soc, 101, 637-655.

.55	.805	.424	.294	.226	.184	.155	.134	.118	.107
	.225	.424	.294	.226	.184	.155	.134	.118	.107
		.168	.294	.226	.184	.155	.134	.118	.107
			.126	.226	.184	.155	.134	.118	.107
				.100	.184	.155	.134	.118	.107
					.084	.155	.134	.118	.107
						.072	.134	.118	.107
							.963	.118	.107
								.056	.107
									.051

Table 1 The matrix g for the case of 10 equally spaced layers for the scheme specified by Eq 13.

1.82	-5.74	9.86	-13.9	17.9	-21.9	25.9	-30.0	34.0	-37.5
	3.92	-9.86	13.9	-17.9	21.9	-25.9	30.0	-34.0	37.5
		5.94	-13.9	17.9	-21.9	25.9	-30.0	34.0	-37.5
			7.96	-17.9	21.9	-25.9	30.0	-34.0	37.5
				9.97	-21.9	25.9	-30.0	34.0	-37.5
					12.0	-25.9	30.0	-34.0	37.5
						14.0	-30.0	34.0	-37.5
							16.0	-34.0	37.5
								18.0	-37.5
									19.5

Table 2 The inverse g^{-1} of the matrix g in Table 1.

	216		-5636
	216		5636
	221		-5657
	235		5573
T	247	g^{-1}T	-5661
	257		5580
	265		-5650
	272		5551
	278.5		-5655
	284.5		5548

Table 3 The mean temperature T of a standard atmosphere and the vector g^{-1}T where g^{-1} is as in Table 2.

Level	Day 1	Day 2	Day 3	Day 4	Day 5
1	1.022	2.006	3.031	4.012	4.984
2	-.997	-2.015	-3.014	-4.022	-5.010
3	.998	2.033	2.998	4.006	5.038
4	-1.006	-1.984	-3.010	-4.018	-5.035
5	.993	1.995	3.009	4.028	4.987
6	-1.002	-2.013	-2.995	-3.966	-4.995
7	1.007	1.976	2.970	3.996	4.999
8	-.983	-2.009	-3.024	-4.012	-5.013
9	1.032	2.028	3.010	3.981	4.957
10	-.954	-1.928	-2.921	-3.937	-4.956

Table 4 Temperatures from the experiment described in Section 3 where we directly force the spurious mode in the original 2D model. The temperatures are for the column where the heating/cooling had an amplitude of 1 deg/day.

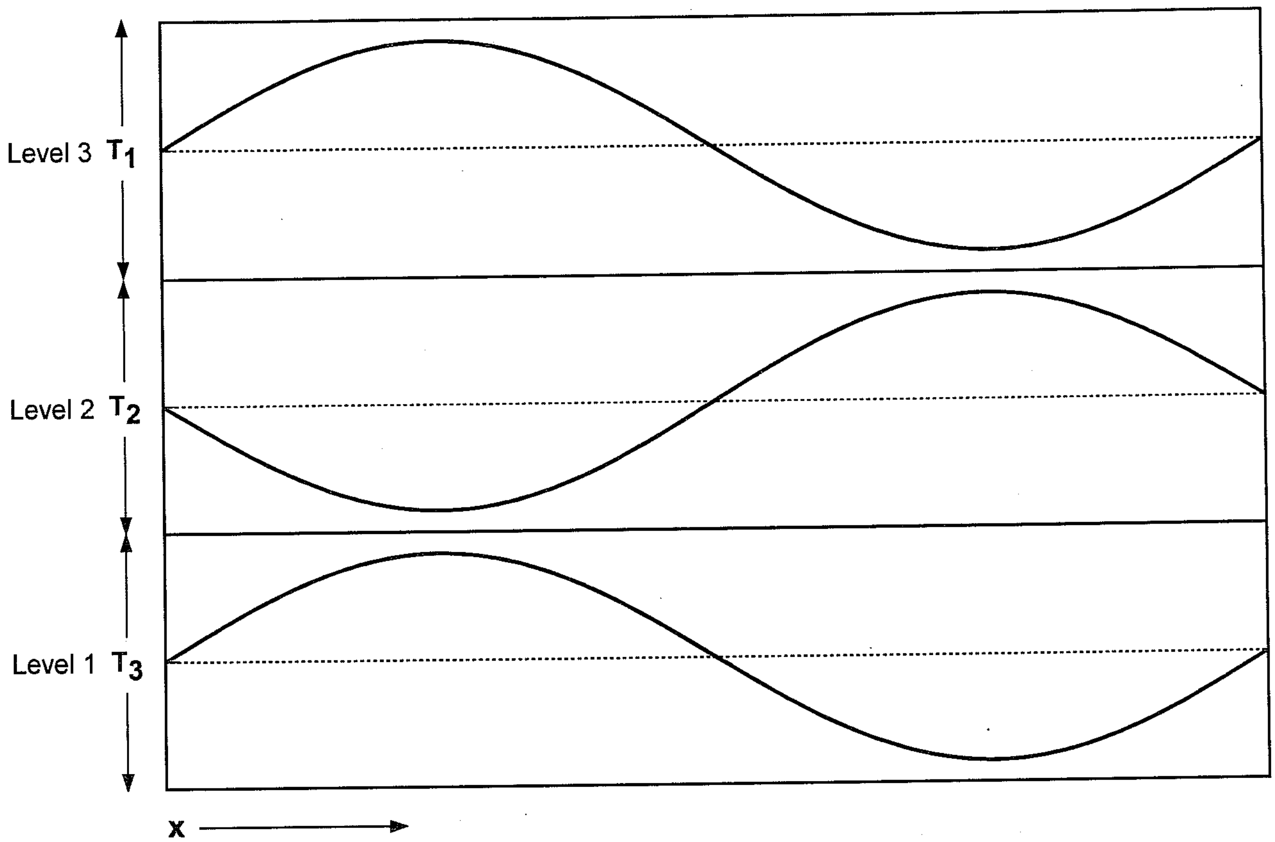

Fig. 1 Schematic of the distribution of temperature in a three-level model which would balance a sinusoidal pressure wave so that the net pressure force at each level is zero.

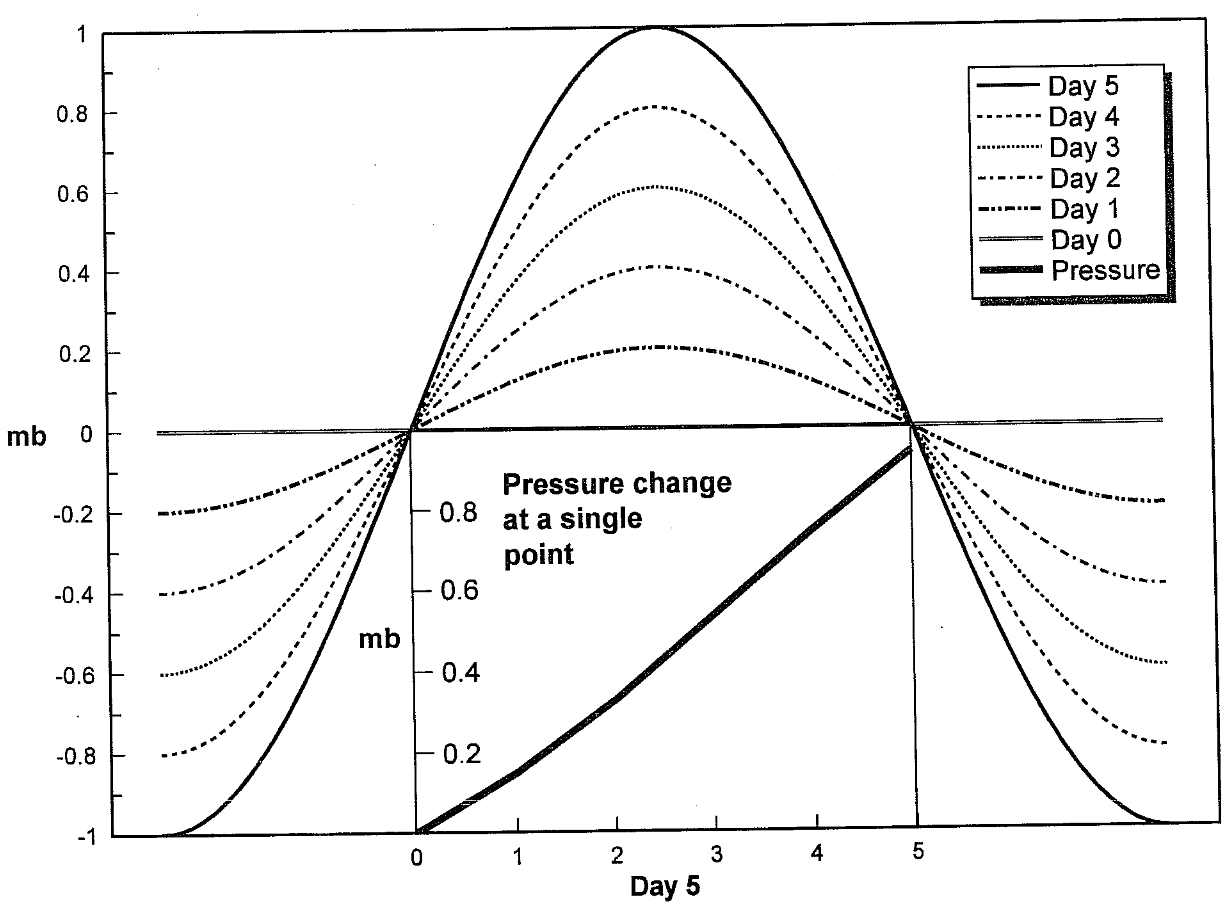

Fig. 2 Behaviour in time of the spatial distribution of surface pressure for the same experiment as Table 4. The inset shows the behaviour of the pressure at the mid-point over the same period.

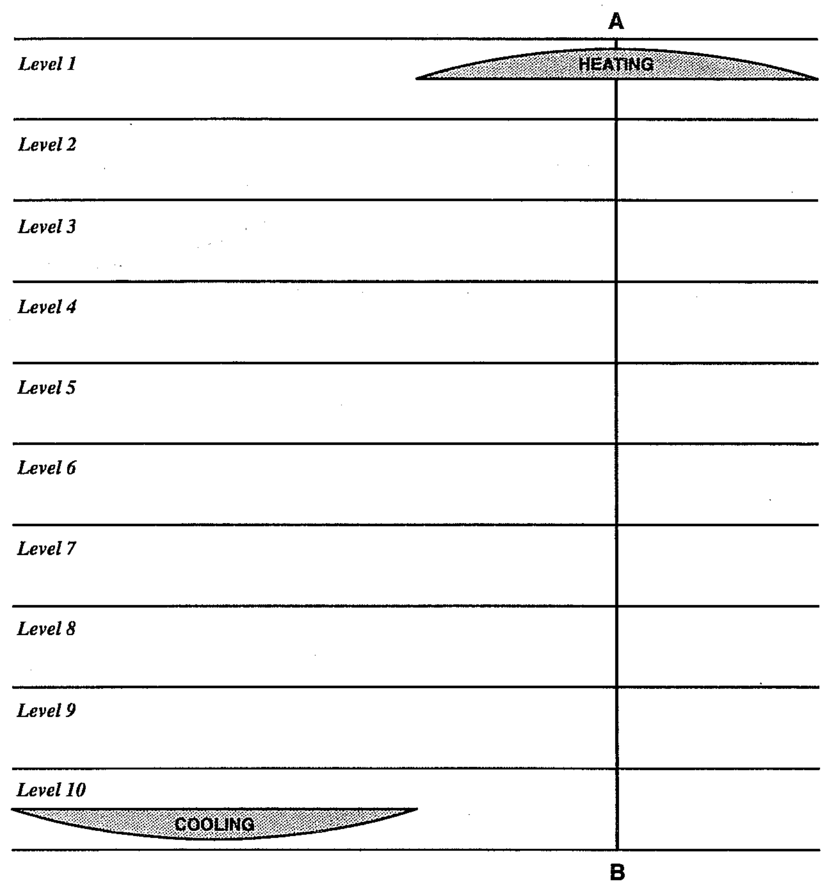

Fig. 3 Schematic of the distribution of heating and cooling in the experiment described in section 4. The column AB is the column for which the perturbation temperatures are plotted in Fig 4.

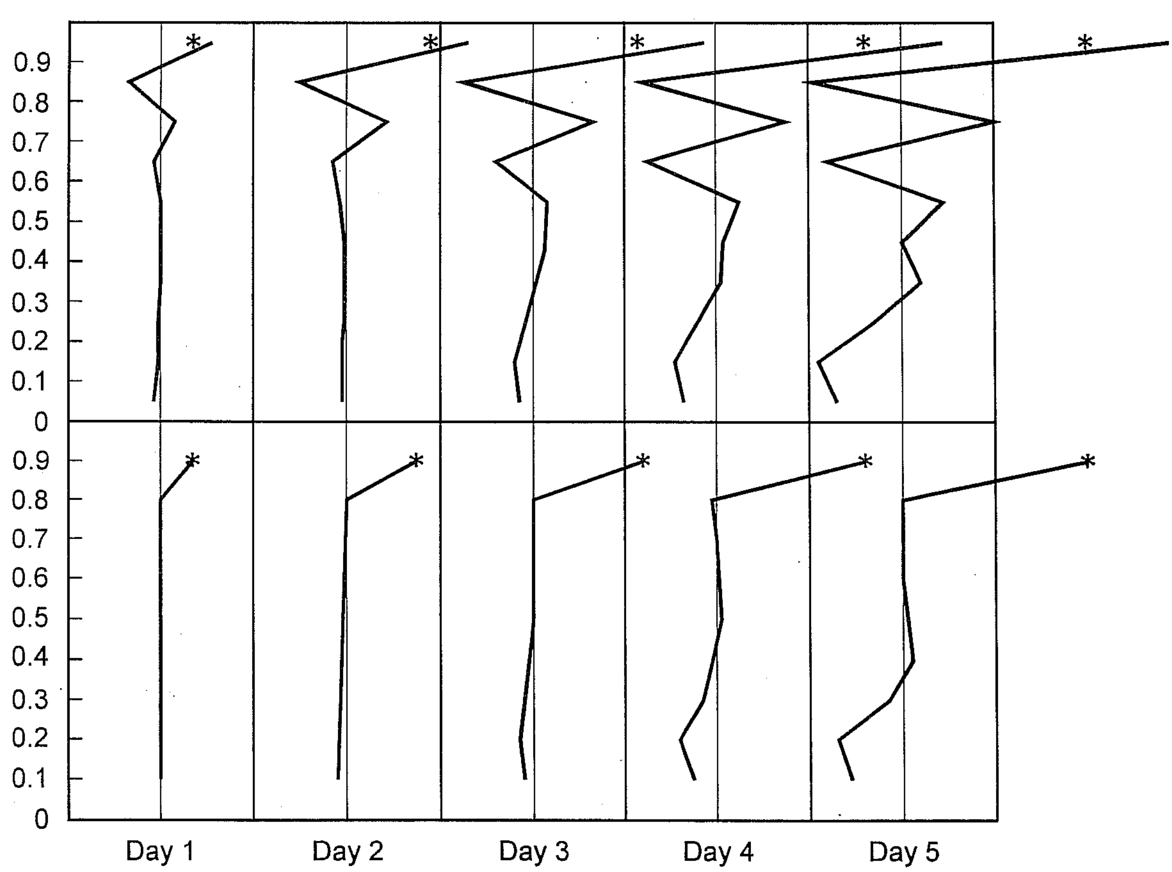

Fig. 4 Time evolution of the perturbation temperature in the column AB of Fig 3 in the experiment described in section 4. The asterisk shows the mean temp-erature in the top layer.