

# Can spectral methods on the sphere live for ever?

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## 1. INTRODUCTION

The current operational deterministic model at ECMWF has a horizontal spectral resolution of T319 (gridlength ~60 km). The 10-year plan envisages the implementation of a deterministic model with a horizontal resolution of about 15 km, or spectral resolution of about T1280, by 2008. In view of the concerns about the efficiency of spectral methods on the sphere at such high resolution and beyond, it is necessary to keep in mind the possibility that a complete change in the method of horizontal discretization may be required. This paper briefly reviews the situation and the prospects for alternative, more efficient, spectral methods on the sphere which might remain competitive for the foreseeable future.

## 2. ADVANTAGES OF SPECTRAL METHODS

Spectral methods on the sphere (based on spherical harmonics) have a number of well-known advantages:

- \* highly accurate computation of horizontal derivatives;
- \* control of aliasing, though this is of less concern in the context of semi-Lagrangian schemes (*Côté and Staniforth, 1988; Hortal, 1999*);
- \* absence of pole problems;
- \* uniform resolution on the sphere;
- \* easy solution of (constant coefficient) elliptic problems (e.g., for semi-implicit schemes);
- \* the spectral method with a reduced Gaussian grid (*Hortal and Simmons, 1991*) is arguably the neatest solution to the problem of providing almost-uniform resolution in gridpoint space on the sphere.

## 3. PROBLEMS AT HIGH RESOLUTION

In the early days of massively parallel processing, it was widely held that spectral methods would run inefficiently on MPP architectures because of their need for global communication. This proved to be much less of a problem than had been feared; relatively simple communica-

tion strategies have proved adequate to ensure efficient scaling on up to 1000 processors (*Barros et al.*, 1995).

A more serious problem is that for a horizontal spectral truncation at total wavenumber  $N$  the number of gridpoint computations per timestep scales as  $O(N^2)$ , the work involved in Fourier transforms scales as  $O(N^2 \log N)$ , while the computation of Legendre transforms scales as  $O(N^3)$ . Thus, it is argued that *eventually* the Legendre transforms will dominate the computation and each doubling of the horizontal resolution will result in an eightfold increase in the computational work per timestep (compared with a fourfold increase for a gridpoint model).

To put this problem in perspective, Table 1 shows the cost of the transforms as a percentage of the work per timestep at the current operational resolution of the ECMWF model, and as measured at two experimental higher resolutions. It is clear that although the relative cost of the Legendre transforms is growing as expected, it will remain bearable beyond T1000. Nevertheless, for horizontal resolutions which might be contemplated in the more distant future, it is evident that the current approach cannot be sustained for ever.

TABLE 1. COST OF TRANSFORMS AS PERCENTAGE OF MODEL TIMESTEP

	Fourier	Legendre
T319	2%	9%
T511	2%	12%
T799	2%	15%

#### 4. POSSIBLE SOLUTIONS

A single Legendre transform can be regarded as a matrix-vector multiplication. Since we are always performing many transforms simultaneously (several variables, all the model levels) the set of transforms can be written as a matrix-matrix multiplication (indeed the transforms in the ECMWF model are coded in this way). This suggests the use of Strassen's fast matrix multiplication algorithm which has asymptotic computational complexity of  $O(N^{2.81})$  and provides useful speedups for matrices of the size considered here (*Bailey*, 1988; *Higham*, 1990). This reduction in computational complexity might be helpful in the medium term, but appears insufficient to save the spectral method for ever.

More promising avenues would be to abandon spherical harmonics and try to use alternative basis functions allowing a known fast transform algorithm (i.e., some form of Fourier series representation), or to find a fast algorithm for the Legendre transforms. We will explore these possibilities in the following sections.

## 5. DOUBLE FOURIER SERIES

The use of double Fourier series on the sphere is motivated not only by the existence of a fast transform algorithm, but also by the close relationship between associated Legendre polynomials and Fourier series. The associated Legendre polynomial with zonal wavenumber  $m$  and total wavenumber  $n$  can be written (Swarztrauber, 1979) as

$$P_n^m(\sin\theta) = \sum_{k=0}^n a_{km} \begin{cases} \cos(k\phi) & m \text{ even} \\ \sin(k\phi) & m \text{ odd} \end{cases}$$

where  $\theta$  is latitude and  $\phi = \theta + \pi/2$ .

Thus, if a function can be represented as a truncated series of associated Legendre functions, it can also be represented by a similarly truncated series of sines or cosines:

$$f^n(\theta) = \sum_{n=m}^N \hat{f}_n^m P_n^m(\sin\theta) = \sum_{k=0}^N \tilde{f}_k^m \begin{cases} \cos(k\phi) \\ \sin(k\phi) \end{cases}$$

Note that the converse is not generally true - for the above equality to hold, the vector of coefficients  $\tilde{f}_k^m$  must be restricted to a *subspace* of dimension  $(N + 1 - m)$ .

The use of double Fourier series leads to accurate horizontal derivatives, fast transforms and easy solution of constant-coefficient elliptic problems. The principal drawback is that pole problems re-emerge unless further steps are taken.

There is a long history of attempts to construct spectral (or pseudospectral) models on the sphere using double Fourier series. Merilees (1973) coded such a pseudospectral shallow-water model, but encountered instability (the return of the pole problem). Orszag (1974) formulated a spectral barotropic vorticity model; no results were shown, but he pointed out the expected stability problem.

More recently, Spatz *et al.* (1998) based a shallow-water model on double Fourier series, stabilizing the computation by using a "spherical harmonic projection". In terms of the above discussion, this projects the vector  $\tilde{f}_k^m$  into the appropriate subspace, so that the model becomes algebraically equivalent to one based on spherical harmonics. The computational complexity of the projection is  $O(N^3)$  per timestep, the same as for the Legendre transforms, though the operation count is somewhat reduced. Cheong (2000) pursued a similar idea ("spherical harmonic filtering") for the barotropic vorticity equation.

*Remarks:*

- (1) The purpose of the filtering/projection is to keep the fields in the correct subspace (the one spanned by the spherical harmonics). It can be done either in gridpoint space or in Fourier space.
- (2) All the above examples are *Eulerian*. It is plausible that the filtering could be dispensed with in a semi-Lagrangian model (especially on a reduced grid).
- (3) Can the projection be done “fast”? An important contribution towards answering this question was made by *Jakob-Chien and Alpert (1997)*; see also the paper by *Spotz* in this volume.

**6. THE ELUSIVE FAST LEGENDRE TRANSFORM (FLT)**

The use of the Fast Fourier Transform (FFT) is so ubiquitous in scientific computation that it was natural to search for a corresponding Fast Legendre Transform (FLT). The history of that search is somewhat chequered! Several breakthroughs have been announced, only to prove premature. Suggested algorithms have turned out to be numerically unstable, to work for the case  $m = 0$  but not for  $m \neq 0$ , or to have a break-even point (compared with the direct algorithm) at a problem size so large as to be useless for our purposes.

There are two basic approaches to deriving FLT algorithms. In the first it is observed that the matrix which relates the Fourier coefficients to the Legendre coefficients has a special structure and can be “compressed”, leading to an approximate algorithm which can be made as accurate as desired (at a cost). An example was the algorithm of *Alpert and Rokhlin (1991)* which was demonstrated for  $m = 0$ , but unfortunately the anticipated generalization to  $m \neq 0$  failed to work. More recent work along these lines was contained in the thesis by *Mohlenkamp (1997)*, but a practical implementation has not (to my knowledge) materialized.

The second approach is to derive an “exact” algorithm based on a divide-and-conquer technique (like that used to derive the FFT). Using Chebyshev transforms and recurrence relations between associated Legendre functions as building blocks, *Driscoll and Healy (1994)* derived a fast Legendre transform algorithm with computational complexity  $O(N \log^2 N)$  per transform (i.e.,  $O(N^2 \log^2 N)$  for the whole model) and with a reasonable break-even point. However, some doubt has been cast on the numerical stability of this algorithm.

The most promising work on the FLT appeared in a recent thesis by *de Inda (2000)*. She implemented a variant of the Driscoll-Healy algorithm (for  $m = 0$ ) which seems to be free of numerical stability problems and to be faster than the direct algorithm for problem sizes as small as  $N = 32$ . However we will have to wait for a generalization to  $m \neq 0$ , and a demonstration of the corresponding inverse transform, before being sure that the FLT problem has finally been solved.

## 7. CONCLUSIONS

The increasing cost of the Legendre transforms means that the spectral method on the sphere as currently implemented cannot remain competitive for ever, though resolutions well in excess of T1000 should be attainable before this becomes a serious problem.

The viability of spectral methods on the sphere could be extended indefinitely, provided that the time-stepping algorithm does not change too radically (see the paper by *Cullen et al.* in this volume), if one or more of the following turns out to work in practice:

- \* double Fourier series without filtering/projection (in a semi-Lagrangian model);
- \* double Fourier series with *fast* filtering/projection;
- \* the Fast Legendre Transform (the recent work of *de Inda* (2000) provides some cause for optimism here).

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