Improved algorithms for computing the non-linear quadruplet wave-wave interactions in deep and shallow water

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1 Introduction

Non-linear wave-wave interactions between pairs of four wave components, so-named quadruplets, play an important role in the evolution of wind generated waves (Young and Van Vledder, 1993). Hasselmann (1962) developed the theoretical framework for these interactions. He formulated an integral expression for the computation of these interactions which is known as the Boltzmann integral for surface gravity waves. Hasselmann found that a set of four waves, called a quadruplet, could exchange energy when the following resonance conditions are satisfied:

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \tag{1}$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4 \tag{2}$$

in which ω_j the radian frequency and $\vec{k}_j$ the wave number (j=1,..,4). The dispersion relation relates the frequency and the wave number:

$$\omega^2 = gk \tanh(kh) . \tag{3}$$

Here, g is the gravitational acceleration and h the water depth. The resonance conditions imply a conservation of energy, action and momentum. Hasselmann (1963) describes the non-linear interactions between wave quadruplets in terms of their action density n, where $n(\vec{k}) = E(\vec{k})/\omega$. The rate of change of action density at a wave number $\vec{k}_1$ due to all quadruplet interactions involving $\vec{k}_1$ is:

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \iiint G(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \times \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ & \times \left[n_1 n_2 (n_3 + n_4) - (n_1 + n_2) n_3 n_4 \right] d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \end{aligned} \tag{4}$$

Where $n_i = n(\vec{k}_i)$ is the action density at wave number $\vec{k}_i$ and G is a complicated coupling coefficient (Herterich and Hasselmann, 1980). The delta functions in (4) ensure that contributions to the integral only occur for quadruplets that satisfy the resonance conditions. The integral expression (4) is also known as the

Boltzmann integral for wind waves. In Russia Zakharov (1968) developed a similar expression, known as the kinetic equation.

The computation of the Boltzmann integral is rather complicated and very time consuming since it requires the solution of a 6-dimensional integral. Because of this it is not feasible to include the full solution of the Boltzmann integral in operational wave models. To that end Hasselmann et al. (1985) developed the Discrete Interaction Approximation which preserves the basic characteristics of the full solution. The development of the DIA triggered the development of third generation wave prediction models, like the WAM model (WAMDI, 1988), WAVEWATCH (Tolman, 1991) and the SWAN model (Booij et al., 1999).

The last years it has become evident that the DIA is not able to properly represent the non-linear transfer function and that it distorts the source term balance in a wind wave spectrum. This is illustrated in Figure 1, which is based on a mean JONSWAP spectrum with f_p=0.4 Hz and α=0.0175.

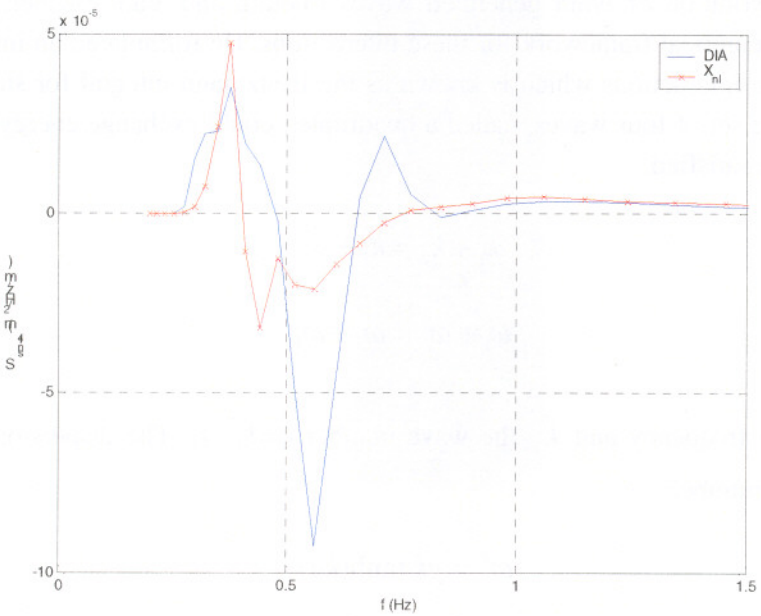

Figure 1: Comparison of non-linear transfer rate computed with an exact method (line with crosses) and with the DIA (solid line) for a mean JONSWAP spectrum with fp=0.4 Hz and α=0.0175.

To overcome this shortcoming of the DIA, WAM-type models are heavily tuned to compensate for the mismatch in the DIA. As a consequence, the further development of source terms for other physical processes is useless as long as the DIA is part of a wave model that is used for such purposes.

In shallow water the non-linear interactions increase in magnitude. This has been shown by Hasselmann and Hasselmann (1981) using the EXACT-NL model. This behaviour is included in WAM-type models by means of a parameterised scaling law (WAMDI, 1988). A careful analysis of depth effects on the non-linear interaction terms, however, shows that this scaling is rather crude. This is because it does not account for changes in the 2d-shape of the non-linear transfer term in shallow water, but also because the parameterisation is based on computations with (deep water) JONSWAP spectra. This is illustrated in Figure 2. For shallow water applications the scaling law either needs to be improved or replaced by a finite depth DIA.

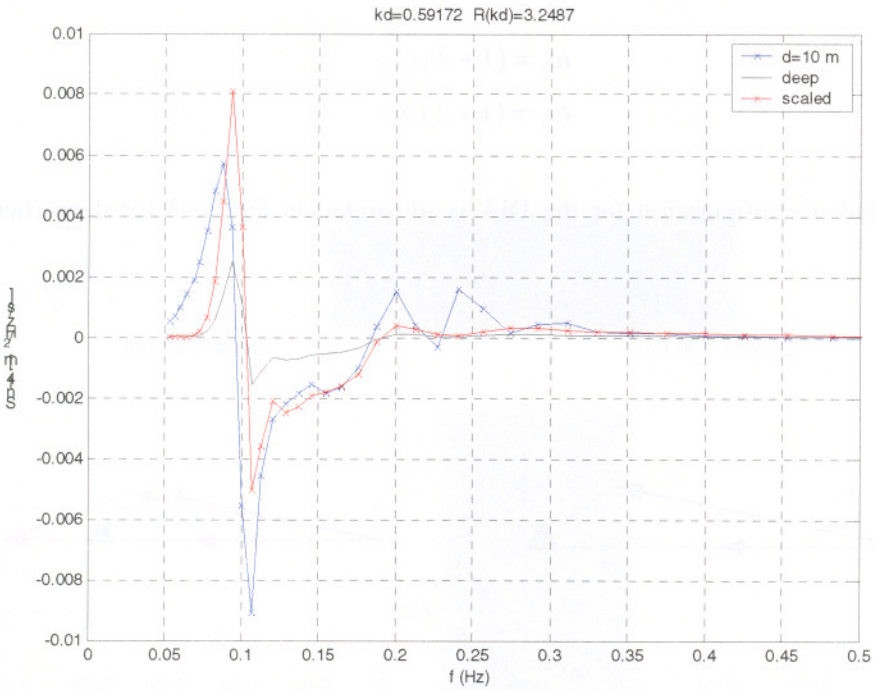

Figure 2: Nonlinear transfer for a JONSWAP spectrum with fp=0.1 Hz and a water depth of 10 m. Deep water transfer, finite depth transfer and scaled deep water transfer.

Another point of concern is the computational requirement of the DIA in operational wave models. In comparison with other source terms it takes most of the required CPU. The relative amount spent in the DIA also depends on the complexity of the numerical scheme of the host model. In the WAM model the DIA takes about 50% of the time whereas in the SWAN model it only takes about 10% of the total time.

To improve the model performance of third generation wave prediction models, at least from the viewpoint of quadruplet wave-wave interactions, attention should be paid to the improvement of the non-linear quadruplet interaction source term and to the development of faster routines for the calculation of these interactions. These issues will be addressed in the following Chapters. First, some developments in the improvement of the DIA will be addressed, followed by an overview of methods for the faster computation of the non-linear interactions.

2 Extending the DIA

In the Classic DIA, developed by Hasselmann et al. (1985), two wave numbers in a wave number quadruplet are equal and the other have different magnitudes and directions. They are related as:

$$\vec{k}_1 = \vec{k}_2 \tag{5}$$

and the frequencies are related as

$$\left.\begin{aligned}\omega_1 &= \omega_2 = \omega \\ \omega_3 &= (1+\lambda)\,\omega \\ \omega_4 &= (1-\lambda)\,\omega\end{aligned}\right\} \tag{6}$$

The typical wave number configuration for the DIA is illustrated in Figure 3 for deep (here 100 m) and shallow water.

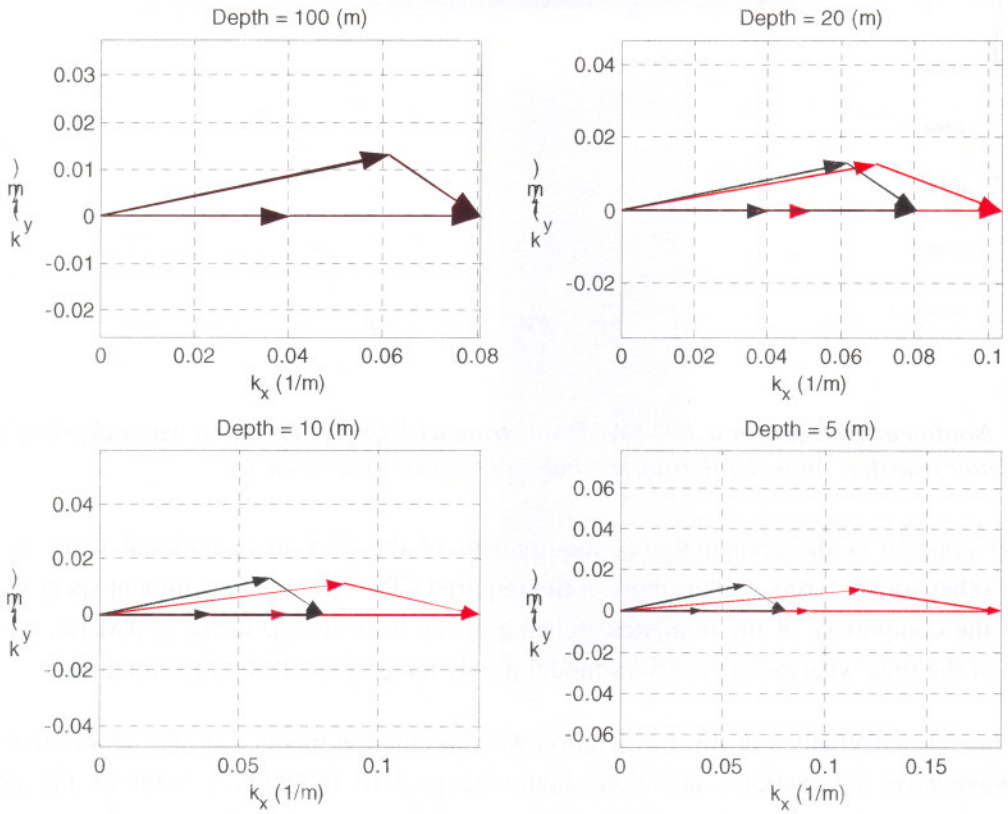

Figure 3: Typical wave number configurations of the DIA in deep (upper left panel) and shallow water.

The basic expression of the DIA is given by:

$$\begin{pmatrix} S_{nl} \\ S_{nl}^{+} \\ S_{nl}^{-} \end{pmatrix} = \begin{pmatrix} -2\dfrac{\Delta f}{\Delta f} \\ (1+\lambda)\dfrac{\Delta f}{\Delta f^{+}} \\ (1-\lambda)\dfrac{\Delta f}{\Delta f^{-}} \end{pmatrix} C_{nl4}\, g^{-4} f^{11} \times \left[E^2 \left(\frac{E^{+}}{(1+\lambda)^4} + \frac{E^{-}}{(1-\lambda)^4} \right) - 2E \frac{E^{+}}{(1+\lambda)^4} \frac{E^{-}}{(1-\lambda)^4} \right] \tag{7}$$

In this expression E, E^- and E^+ are the energy densities at the interacting wave numbers and S_{nl}, S_{nl}^- and S_{nl}^+ are the corresponding rates of change of energy density due to the non-linear transfer in a quadruplet. To estimate the transfer in a wave spectrum, expression (7) is applied to all discrete spectral bins of a spectrum in which f and E are taken as the central bin.

In the case of a geometric sequence of frequencies the frequency step is proportional to the associated frequency $\Delta f \sim f$. Since

$$\begin{aligned} \Delta f^+ &= (1+\lambda)\Delta f \\ \Delta f^- &= (1-\lambda)\Delta f \end{aligned} \tag{8}$$

Using (8) equation (7) simplifies to:

$$\begin{pmatrix} S_{nl} \\ S_{nl}^+ \\ S_{nl}^- \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} C'_{nl4} g^{-4} f^{11} \left[E^2 \left(\frac{E^+}{(1+\lambda)^4} + \frac{E^-}{(1-\lambda)^4} \right) - 2\frac{EE^+E^-}{(1-\lambda^2)^4} \right] \tag{9}$$

In the WAM and SWAN model, $\lambda = 0.25$ and $C_{nl4} = 3\times10^{-7}$, whereas in the WAVEWATCH III model (Tolman and Chalikov, 1996) $C_{nl4} = 1\times10^{-7}$ to avoid overestimation of the transfer towards higher frequencies. In Japan a value of $\lambda = 0.19$ is preferred (Hashimoto, 1999) since the DIA with this value gives a better agreement with exact solutions for JONSWAP spectra.

2.1 The Multiple DIA

The Classic DIA can be extended in a variety of ways. The most simple way is to add wave number configurations with different values of λ and C_{nl4}:

$$\begin{pmatrix} S_{nl} \\ S_{nl}^+ \\ S_{nl}^- \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \sum_i C_{nl4,i} g^{-4} f^{11} \left[E^2 \left(\frac{E^+}{(1+\lambda_i)^4} + \frac{E^-}{(1-\lambda_i)^4} \right) - 2\frac{EE^+E^-}{(1-\lambda_i^2)^4} \right] \tag{10}$$

Such extensions have been proposed by Hashimoto (1999) and by Van Vledder et al. (2000). This approach is limited because the shapes of the resulting transfer function have insufficient degrees of freedom to approximate the exact solution.

2.2 The Generalized DIA

A more general way to extend the DIA is deviate from the assumption $\vec{k}_1 = \vec{k}_2$. An example of such a wave number configuration is shown in Figure 4. To that end the following general relationship is proposed for the four wave numbers in a basic configuration:

$$\begin{aligned}\omega_1 &= \omega \\ \omega_2 &= (1+\mu)\omega = \omega^* \\ \omega_3 &= (1+\lambda)\omega = \omega^+ \\ \omega_4 &= (1-\lambda-\mu)\omega = \omega^-\end{aligned} \tag{11}$$

and

$$\theta_2 = \theta_1 + \Delta\theta \tag{12}$$

in which $\Delta\theta$ is the angle between the wave numbers $\vec{k}_1$ and $\vec{k}_2$. This leads to the following expression of a GDIA:

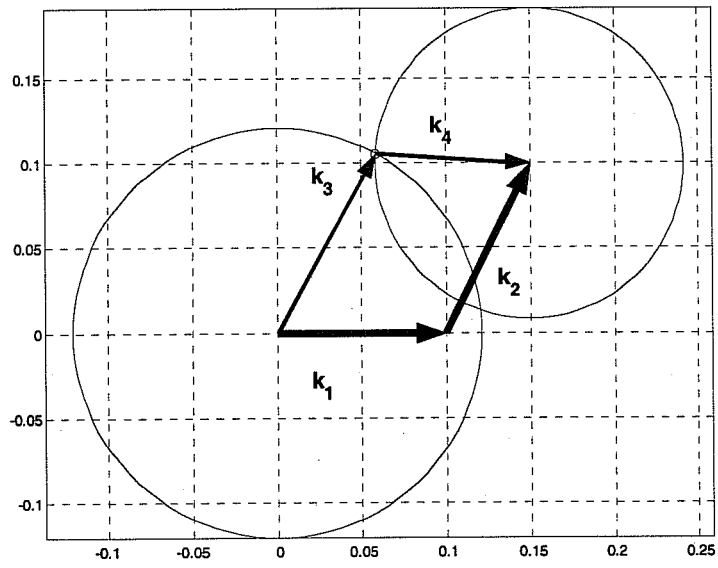

Figure 4: Example of a wave number configuration of the GDIA. The circles are used to solve the resonance conditions in a geometric way.

$$\begin{pmatrix}\delta S_{nl} \\ \delta S_{nl}^* \\ \delta S_{nl}^+ \\ \delta S_{nl}^-\end{pmatrix} = \begin{pmatrix} -1\dfrac{\Delta f}{\Delta f} \\ -(1+\mu)\dfrac{\Delta f}{\Delta f^*} \\ (1+\lambda)\dfrac{\Delta f}{\Delta f^+} \\ (1-\lambda-\mu)\dfrac{\Delta f}{\Delta f^-}\end{pmatrix} C_{nl4} g^{-4} f^{11} \times \tag{13}$$

$$\left\{\frac{EE^*}{(1+\mu)^4}\left(\frac{E^+}{(1+\lambda)^4}+\frac{E^-}{(1-\lambda-\mu)^4}\right)-\left(E+\frac{E^+}{(1+\mu)^4}\right)\frac{E^*}{(1+\lambda)^4}\frac{E^-}{(1-\lambda-\mu)^4}\right\}$$

The parameters λ, μ and $\Delta\theta$ cannot be chose at random, they are bounded by the fact that they should give valid solutions of the resonance conditions. Of course, other kinds of relationships between the four wave numbers in a wave number configuration can be given, e.g. see Masuda (1980). For a geometric spacing of the frequencies we have, expression (13) can be simplified such that the first term on the right hand side of (13) reduces to $[-1\ -1\ 1\ 1]^T$.

The general expression (13) can be used as a basic type of DIA which can be extended to a multiple GDIA by considering a finite sequence of configurations specified by the parameters λ_i, μ_i , $C_{nl4,i}$ and $\Delta\theta_i$. The GDIA is able to construct a set of 'orthogonal' basis functions to approximate the exact form quite well, which in the limiting case of many wave number configurations approaches the exact solution. Details of the GDIA are given in Van Vledder (2001b).

2.3 Finite depth DIA

Finite depth wave modeling is important for many coastal wave applications. As described in Herterich and Hasselmann (1980) the transfer rate increases as the wave enter shallow water. Since finite depth wave-wave interactions are relatively more difficult to compute a depth scaling has been proposed for the nonlinear quadruplet interactions. In the WAM paper (WAMDI, 1988) a scaling is proposed which is based on the work of Herterich and Hasselmann (1980), which comprises a scale factor equal for all spectral bins (in contrast to a k-dependence as suggested in the WAM paper, Eq. 3.7). This factor R is a function of the dimensionless wave number $\overline{kd}$ and is given by:

$$R(x) = 1 + \frac{5.5}{x}\left(1 - \frac{5x}{6}\right)\exp\left(-\frac{5x}{4}\right) \tag{14}$$

With $x = \frac{3}{4}\overline{kd}$ and in which $\overline{k}$ is a mean wave number. This scaling does not change the shape of the nonlinear transfer. For $\overline{kd} < 0.8$ this assumption does not hold, as can be seen in Figure 2 but also already in Fig. 2 of Hasselmann and Hasselmann (1985a). The most striking feature is that the first positive lobe shifts towards lower frequencies and becomes wider than the scaled deep water transfer. This deviation from the scaling law is important for the estimation of the rate of frequency downshifting of swell waves in shallow water. As such, the WAM depth scaling may contribute to the under-prediction of swell wave periods in, e.g. the Dutch, coastal waters. The computation of the finite depth transfer in WAM type models can be improved by omitting the WAM depth scaling by a direct computation of the transfer rate with a finite depth version of the DIA. This has been achieved by re-deriving the DIA while keeping all finite depth terms (cf. Van Vledder and Pihl, 2001).

Following Hasselmann and Hasselmann (1981) and Rasmussen (1995) equation (4) can be transformed to:

$$\begin{pmatrix} \Delta n_1 \\ \Delta n_2 \\ \Delta n_3 \\ \Delta n_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \frac{\sigma}{4} |J| \times P \times \Delta k_s \Delta\theta_s \Delta\omega_s \Delta\omega_1 \Delta\omega_3 \Delta t \tag{15}$$

in which $\vec{k}_s = \vec{k}_1 + \vec{k}_2$, $\omega_s = \omega_1 + \omega_2$ and J the Jacobean of the transformation. Equation (15) refers to the principle of detailed balance in which the rate of action in all four interacting bins is equal. This feature, together with additional symmetries is exploited in the program EXACT-NL developed by Hasselmann and Hasselmann (1985b).

The Jacobean J in Eq. (15) is given as (Hasselmann and Hasselmann, 1981 and Rasmussen, 1995):

$$J = \frac{k_s}{c_{g1} c_{g2} c_{g3} c_{g4} \sin(\theta_2 - \theta_1) \sin(\theta_4 - \theta_3)} \tag{16}$$

The product term is given by:

$$P = n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2) \tag{17}$$

Using various transformations, the rate of change of wave action can be transformed in the rate of change of energy density in the four bins involved in a quadruplet wave-wave interaction. Equation (16) contains a singularity for the DIA assumption of $\vec{k}_1 = \vec{k}_2$ This singularity, however, can easily be removed (Pihl, 2001, Van Vledder and Pihl, 2001). For a geometric frequency spacing the following expression is obtained:

$$\begin{pmatrix} S_{nl} \\ S_{nl}^{+} \\ S_{nl}^{-} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} C_{nl4} G(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \frac{k^2}{c_g^2 c_g^+ c_g^-} f^3 \times \left[\left(\frac{c_g E}{f k} \right)^2 \left\{ \left(\frac{c_g^+ E^+}{f^+ k^+} \right) + \left(\frac{c_g^- E^-}{f^- k^-} \right) \right\} - 2 \left(\frac{c_g E}{f k} \right) \left(\frac{c_g^+ E^+}{f^+ k^+} \right) \left(\frac{c^- E^-}{f^- k^-} \right) \right] \tag{18}$$

A crucial difference with the deep water DIA is that the group velocities $c_{g,i}$, the wave numbers k_i and the interaction coefficient G appear in this expression. Another feature of the SDIA is that the shape of the wave number configuration changes. This is illustrated in Figure 3. This feature implies that, depending on the water depth, different bins interact in a discrete spectrum. For deep water, these variables can be replaced by frequencies and equation (18) reduces to expression (9).

Combining all the possible extensions to the DIA, a Multiple Generalized Finite Depth Discrete Interaction Approximation can be defined which should approximate the exact transfer in the limit of many wave number configurations.

The key problem is the determination of the coefficients λ_i, μ_i, $\Delta\theta_i$ and $C_{nl4,i}$ of a multiple Generalised DIA. Various approaches can be followed to obtain these coefficients for deep and shallow water. Essential elements are the use of a large set of test directional spectra with a variety of characteristics and the use of a benchmark model to compute the non-linear interactions with an exact method. One dataset will be used to determine the coefficients and another (independent) dataset will be used to verify the extended DIA.

One of the techniques for the determination of the coefficients is to treat it as a linear problem and to determine only the coefficients of proportionality by solving a least-squares (LSQ) problem. This can be achieved by using Householder transformations of matrices which are filled with results of exact and approximate calculations of the non-linear transfer for a set of test spectra Hashimoto (1999). In his approach, however, the parameters (λ_i) defining the wave number configuration were pre-set and were not direct part of the calibration process. This method resulted in positive and negative coefficients of proportionality. These negative coefficients, however, are unphysical and limit the applicability of the multiple DIA (MDIA) for spectra that are not similar in shape compared to the spectra used for derivation of his MDIA. Still, the LSQ provides a good starting point for the determination of the coefficients of a multiple DIA. Improvements can be made by using unequal weights in defining the LSQ sum and by imposing constraints on the coefficients.

A disadvantage of computing the coefficients of proportionality with the previously described rigorous numerical method is that the inherent relative contribution of each individual wave number configuration is not accounted for. If these inter-relationships can be preserved, the optimisation procedure reduces to the estimation of only one coefficient. This has major benefits, in the sense that the numerical requirements for the LSQ problem reduce significantly. This approach is similar to the re-normalisation technique of Snyder et al. (1993).

A crucial step in this process is the choice of the test spectra. The range of spectral characteristics must be wide enough to ensure a wide range of applicability of the approximate methods. In this sense there is a similarity with the training of neural network techniques.

The inherent relative contribution can be preserved by downscaling exact methods. Treating the WRT method (Tracy and Resio, 1982, Resio and Perrie, 1991) as a mathematical integration problem, down-scaling is similar to increasing the step size in the evaluation of integrals and provides an alternative approach to derive Multiple General Discrete Interaction Approximations, both for deep and shallow water.

3 Filtering methods

Filtering methods can be used to limit the number of calculations from the integration space. This is easily achieved by omitting interactions that exchange only a small (below a certain threshold) amount of energy or action. This feature is exploited in the EXACT-NL model (Hasselmann and Hasselmann, 1981). This model uses a reference spectrum to determine the part of the interaction space that contributes significantly to this transfer. Minor contributions are filtered out. Experience shows that 10% of the interaction space contributes for about 95% to the total non-linear transfer in the wave number regions with significant transfer. The filtered interaction space can then be used for similar spectra. If a spectrum deviates too much from the reference spectrum, the filtered interaction space must be re-computed. In practice, similarly shaped spectra are mapped on the reference spectrum by a shift along the frequency axis and/or by a rotation over a certain angle. Then the non-linear transfer is computed using the transformed spectrum and the resulting transfer is mapped back by inverse transformations. The EXACT-NL method has been used by e.g. Van Vledder (1990), Weber (1987).

Another filtering method can be applied during the actual computation of the non-linear transfer. During run-time the total amount of energy at the central bin, e.g. in the DIA, is compared to some threshold value. If the energy is below this threshold value, the discrete interaction for the wave number configuration containing

this central bin can be skipped, thereby saving a certain amount of CPU. The threshold value can be set to a certain fraction of the total amount of energy in a spectrum. This method can also be applied to exact methods, e.g. in the outer loop of the WRT method. A disadvantage of such methods is that is may be incompatible with vector or parallel processing on certain computers. On scalar machines it may give a speed-up of a factor 4, especially for spectra defined on a full circle and in which the spectrum is located in a small part of the spectral grid.

3.1 Sector and symmetric grids

Another method to limit the integration space is to define a spectrum over a certain sector, for instance from –120° till 120°, optionally in combination with the assumption of symmetry along the zero direction axis. These methods are only useful for research purposes to study the evolution of the energy balance in 1d-models which integrate the action balance equation either with respect to time or fetch.

3.2 Spectral resolution

Since the computation of the non-linear interactions includes triple products of energy densities, the computational requirements thereof roughly scale with the third power of the number of spectral bins. It is therefore essential to optimise the spectral resolution.

With respect to the frequency resolution to following comments can be made. The DIA was derived for a geometric frequency resolution of 10%, i.e. $f_{i+1} = 1.1 \times f_i$. Taking a higher resolution gives erratic results as was shown by van Vledder et al. (2000). Taking a lower resolution gives unrealistic results. Higher resolutions can only be used if the DIA is replaced by a method which also includes shorter range interactions. This can be achieved by exact methods or by a Multiple GDIA. With respect to the directional resolution a value of 10° is optimal for practical use in operational wave prediction models. A finer directional resolution is only advised if one wants to compute the non-linear transfer for very narrow swell spectra.

4 Efficient algorithms

4.1 Integration schemes

The WAM and WAVEWATCH model use the quasi-implicit integration scheme for the integration of the action balance equation. A key element of this scheme is the diagonal term, defined as:

$$\Lambda_{nl}(f,\theta) = \frac{\partial S_{nl}(f,\theta)}{\partial E(f,\theta)} \tag{19}$$

Application of this operator on the expression for the DIA gives:

$$\partial\Lambda_{nl} = 2C_{nl4} g^{-4} f^{11} \left[2E \left(\frac{E^+}{(1+\lambda)^4} + \frac{E^-}{(1-\lambda)^4} \right) - \frac{2E^+E^-}{(1-\lambda^2)^4} \right] \tag{20}$$

$$\partial\Lambda_{nl}^{+} = -C_{nl4} g^{-4} f^{11} \left[\frac{E^2}{(1+\lambda)^4} - \frac{2EE^{-}}{(1-\lambda^2)^4} \right] \tag{21}$$

$$\partial\Lambda_{nl}^{-} = -C_{nl4} g^{-4} f^{11} \left[\frac{E^2}{(1-\lambda)^4} - \frac{2EE^{+}}{(1-\lambda^2)^4} \right] \tag{22}$$

4.2 The diagonal term for the GDIA

The diagonal terms for the GDIA are computed from the general expression of the GDIA as follows:

$$\partial\Lambda_{nl} = C g^{-4} f^{11} \left[\frac{E^*}{(1+\mu)^4} \left(\frac{E^+}{(1+\lambda)^4} + \frac{E^-}{(1-\lambda-\mu)^4} \right) - \frac{E^+}{(1+\lambda)^4} \frac{E^-}{(1-\lambda-\mu)^4} \right] \tag{23}$$

$$\partial\Lambda_{nl}^{*} = C g^{-4} f^{11} \left[\frac{E}{(1+\mu)^4} \left(\frac{E^+}{(1+\lambda)^4} + \frac{E^-}{(1-\lambda-\mu)^4} \right) - \frac{1}{(1+\mu)^4} \frac{E^+}{(1+\lambda)^4} \frac{E^-}{(1-\lambda-\mu)^4} \right] \tag{24}$$

$$\partial\Lambda_{nl}^{+} = C g^{-4} f^{11} \left[\frac{EE^*}{(1+\mu)^4} \frac{1}{(1+\lambda)^4} - \left(E + \frac{E^*}{(1+\mu)^4} \right) \frac{1}{(1+\lambda)^4} \frac{E^-}{(1-\lambda-\mu)^4} \right] \tag{25}$$

$$\partial\Lambda_{nl}^{-} = C g^{-4} f^{11} \left[\frac{EE^*}{(1+\mu)^4} \frac{1}{(1-\lambda-\mu)^4} - \left(E + \frac{E^*}{(1+\mu)^4} \right) \frac{E^+}{(1+\lambda)^4} \frac{1}{(1-\lambda-\mu)^4} \right] \tag{26}$$

In the same way the diagonal terms for the finite depth DIA can be derived. It is also possible to derive a diagonal term for exact methods. This will be illustrated for the Webb/Resio/Tracy method. The WRT method for solving the Boltzmann integral (4) can be written in the form:

$$\frac{\partial n_1}{\partial t} = \int d\vec{k}_3 T\left(\vec{k}_1, \vec{k}_3\right) \tag{27}$$

in which the function T is given by:

$$T\left(\vec{k}_1, \vec{k}_3\right) = \oint_{\vec{s}} G \times \left| \frac{\partial W(\vec{s}, \vec{n})}{\partial \vec{n}} \right|^{-1} \times \left[n_1 n_3 \left(n_4 - n_2\right) + n_2 n_4 \left(n_3 - n_1\right) \right] d\vec{s} \tag{28}$$

Here, G is the coupling coefficient for the non-linear transfer and W is the gradient term which arises by integration over a delta-function. In practise equation (28) consists of the integration along a closed locus (Tracy and Resio, 1982; Resio and Perrie, 1991, and Van Vledder 2000).

The diagonal term for the WRT method is easily obtained from (28) and is given for the symmetric contribution to the transfer of n_1 and n_3:

$$\Lambda_1 = \frac{\partial T\left(\vec{k}_1, \vec{k}_3\right)}{\partial n_1} = \oint_{\vec{s}} G \times \left| \frac{\partial W\left(\vec{s}, \vec{n}\right)}{\partial \vec{n}} \right|^{-1} \times \left[n_3 \left(n_4 - n_2 \right) + n_2 n_3 n_4 \right] d\vec{s} \tag{29}$$

$$\Lambda_3 = \frac{\partial T\left(\vec{k}_1, \vec{k}_3\right)}{\partial n_3} = \oint_{\vec{s}} G \times \left| \frac{\partial W\left(\vec{s}, \vec{n}\right)}{\partial \vec{n}} \right|^{-1} \times \left[n_1 \left(n_4 - n_2 \right) - n_1 n_2 n_4 \right] d\vec{s} \tag{30}$$

These expressions have been included in the WRT method by Van Vledder (2001a).

4.3 The triplet method

In the DIA the energy densities E, E^- and E^+ are computed from the discrete energy density bins by means of linear interpolation from neighbouring bins:

$$E^- = \sum_{i=1}^{4} v_i E_i^- \tag{31}$$

and

$$E^+ = \sum_{i=1}^{4} w_i E_i^+ \tag{32}$$

in which v_i and w_i are the weights to obtain the energy density from the 4 surrounding points E_i^- and E_i^+. By definition the weights satisfy:

$$\sum_{i=1}^{4} v_i = 1 \tag{33}$$

$$\sum_{i=1}^{4} w_i = 1 \tag{34}$$

Substitution of the expressions (31) and (32) into the general expression for the DIA (10) yields:

$$\begin{pmatrix} \delta S_{nl} \\ \delta S_{nl}^{+} \\ \delta S_{nl}^{-} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} Cg^{-4} f^{11} \left[E^2 \left(\frac{\sum_{i=1}^{4} w_i E_i^{+}}{(1+\lambda)^4} + \frac{\sum_{i=1}^{4} v_i E_i^{-}}{(1-\lambda)^4} \right) - \frac{2E\left(\sum_{i=1}^{4} w_i E_i^{+} \right)\left(\sum_{i=1}^{4} v_i E_i^{-} \right)}{(1-\lambda^2)^4} \right] \tag{35}$$

Elaboration of the term between the square brackets gives an expression in the form of a finite sum of triplets (i.e. products of three energy densities at discrete spectral grid points):

$$\sum_{j=1}^{N} \alpha_j E_{p(j)} E_{q(j)} E_{r(j)} \tag{36}$$

in which the terms $p(j)$, $q(j)$ and $r(j)$ refer to all interacting bins in a triplet, and j refers to a spectral bin in a 2-d spectrum. For the DIA the value of N is equal to 26, which doubles to 52 if the mirror image is taken into account. So, the change in energy density by the non-linear interactions can be written as:

$$\begin{pmatrix} \delta S_{nl} \\ \delta S_{nl}^{+} \\ \delta S_{nl}^{-} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} Cg^{-4} f^{11} \sum_{j=1}^{N} \alpha_j E_{p(j)} E_{q(j)} E_{r(j)} \tag{37}$$

The resulting change in energy density within an interacting quadruplet is redistributed among the surrounding bins using the same weights as used for the determination of the energy densities:

$$\delta S_{nl,i}^{-} = v_i \times \delta S_{nl}^{-} \tag{38}$$

$$\delta S_{nl,i}^{+} = w_i \times \delta S_{nl}^{+} \tag{39}$$

Finally, combination of all expressions results in a transfer for an arbitrary spectral bin with reference index $s(j)$ which can be written as a finite number of triple products of energy densities in discrete spectral grid points:

$$S_{nl4,s(k)} = \sum_{k} \delta_k E_{p(k)} E_{q(k)} E_{r(k)} \tag{40}$$

The problem is reduced to the determination of all indices and weights. This is achieved by mean of a dedicated program which resembles an algebraic manipulator. In this process all contributions with equal indices for p(k), q(k), r(k) and s(k) are combined by adding their weights δ(k). In this way all possible symmetries are automatically accounted for. The weights in expression (40) need only to be computed for one direction sector, since the weights for all other directions can be obtained by a simple rotation. The triplet method also offers the possibility to filter out small contributions to the total transfer on the basis of the weights δ(k). All remaining weights and indices of the triplets can be stored in memory or on disk.

The number of triplets can be greatly reduced by assuming a piece-wise constant representation of the spectrum in the same way as Snyder et al. (1993). Using this assumption the bi-linear interpolation of the

Eqs. (31) and (32) is replaced by using a single value corresponding to the nearest bin. The effect on a discrete spectrum is shown in Figure 5. It is to be discussed and analysed if this approach gives valid transfer rates for the higher frequencies which are believed to follow a power law and not a staircase decrease.

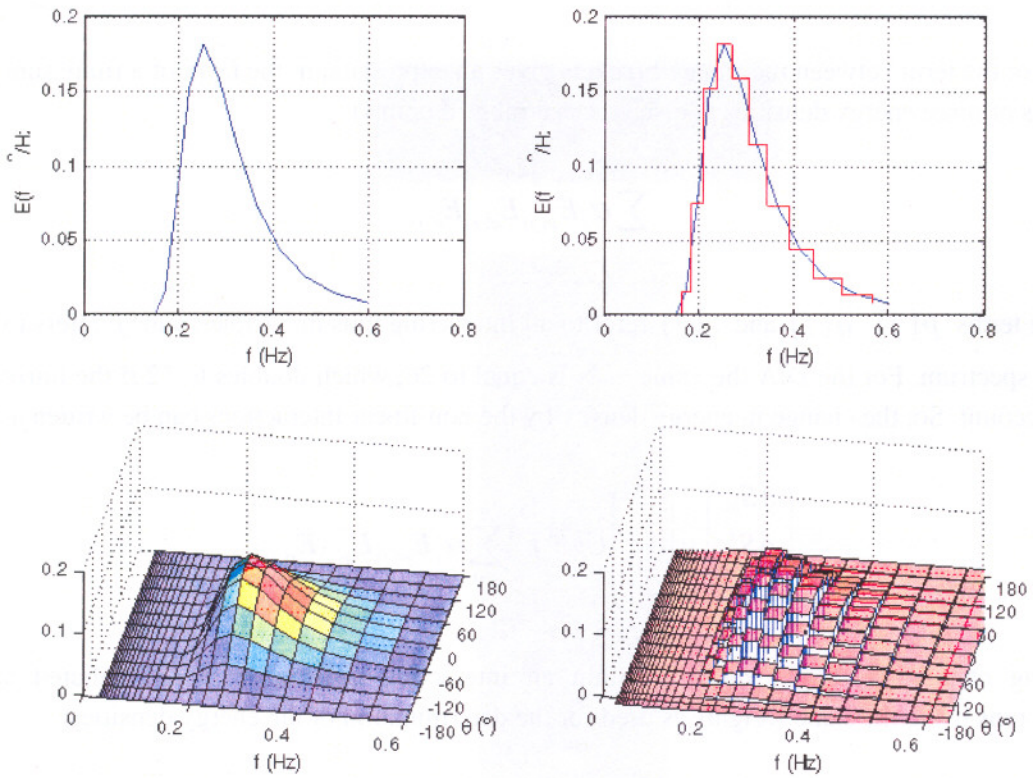

Figure 5: Bi-linear and piece-wise constant representation of a discretized wave spectrum

Since there are much more possible triplets than bins in a discrete spectrum it is more efficient to evaluate expression (40) using a nested loop over the triplets. To that end the indices of all triplets are sorted such that $p(k) \leq q(k) \leq r(k)$, followed by a factorisation. In this way the outer loop of the algorithm is over all $p(k)$, the next loop is over all $q(k)$, the next loop over $r(k)$ and the most inner loop is over all $\delta(k)$ and $s(k)$. The main advantage of the triplet method is that all weights and indices, including the combining, filtering, sorting and factorisation can be computed and stored outside the wave model. The number of lines of code for this algorithm is about 20, except for I/O related operations to retrieve all weight and indices of the triplets.

The weights and indices of the triplet method can also be derived for the Generalized DIA, or Finite depth DIA, and exact methods like the Webb/Resio/Tracy method (Van Vledder, 2001c). For a discretized spectrum the weights and indices depend on the spectral grid, the power of the parameterised spectral tail and the water depth. The triplet method resembles the method derived by Snyder et al. (1993).

Since the triplet method can be derived for fast and inaccurate methods like the DIA and also for time consuming exact methods it can be considered as the extremes of a unified approach to compute the non-linear transfer for a discrete spectrum. In between are extended DIA's or stripped exact methods.

5 Discussion

The present version of the WAM model uses the DIA for the estimation of the non-linear transfer in a wave spectrum. It is now generally accepted that the DIA has many shortcomings and needs to be replaced by more accurate and fast alternatives. The DIA can be improved in a numbers of ways. Firstly, the number of basic wave number configurations should be extended leading to a Multiple DIA (MDIA). Secondly, the shape of the basic wave number configurations should be more flexible. This can be achieved by using the Generalised DIA (GDIA). Thirdly, the depth scaling in shallow water needs to be replaced by using the finite depth or shallow water DIA (SDIA). Combining these extension leads to the Multiple Generalised Shallow water DIA. The determination and validation of the coefficients of such an extended DIA is a major challenge for the coming years. The other way is to strip down and re-normalise exact methods which preserve the relative contribution of each wave number configuration. Essential to the optimisation process is to have the availability of validated methods for the exact computation of these interactions in deep and shallow water, preferably in operational models like WaveWatch, SWAN and WAM. Next, any extended DIA needs to be validated for a number of situations. Not only against carefully selected sets of (academic) target spectra but also in realistic field cases and in quasi-operational environments.

At the same time attention must be given to find efficient algorithms for the computation of the non-linear interactions. The triplet approach suggested in this paper might be a workable method. It is noted here that the basic ideas behind the triplet method can also be applied to compute the triads interactions in a wave spectrum yielding a duplet method.

Any step in improving the physics of a single source initially leads to a poorer performance of a well-tuned wave model and then re-tuning is necessary. This is because, such models have been tuned to compensate for errors in one or more source terms. Replacing the DIA by something much better is feasible and recommended since the exact solution is available to make improvements in the right direction.

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