Conservation issues

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1. Introduction

It is generally agreed that a discretized numerical model should reproduce at least some of the conservation properties possessed by the corresponding continuous equations. An obvious example, of importance in long integrations, is that the total mass of the atmosphere should be conserved. Similarly, the total mass of a tracer such as moisture or a chemical species should be conserved by the purely dynamical (advective) part of the model. Although the maintenance of conservation properties may be less important when there are significant *real* sources and sinks, it may be argued that introducing *spurious* sources and sinks, by failing to respect the properties of the underlying dynamical equations, could easily undermine the realism of the simulation.

At the same time it is worth observing that if the continuous equations conserve a quantity X, then if the numerical scheme is accurate it should conserve X reasonably well; while a scheme that conserves X exactly but is otherwise inaccurate is not very useful.

2. Eulerian vs. semi-Lagrangian schemes

Discussion of conservation properties has become more prominent in recent years as semi-Lagrangian schemes have increasingly replaced Eulerian schemes, first for numerical weather prediction models and more recently for climate models. In an Eulerian model it is generally straightforward to design the discretization so that the total mass of the atmosphere (and that of each tracer quantity) is conserved, simply by considering the fluxes through the edges of grid boxes. On the other hand, in a semi-Lagrangian model the discretization is designed quite differently, and any formal conservation properties are normally lost. The great advantage of semi-Lagrangian integration schemes over their Eulerian counterparts is their computational efficiency, which many modellers are unwilling to forgo. It is therefore of interest to try to restore conservation properties to semi-Lagrangian schemes, thereby combining the merits of Eulerian and semi-Lagrangian models.

There are basically two approaches to tackle the problem. The first is to apply an "*a posteriori* fix", typically after each timestep. The second (and more elegant) approach is to modify the semi-Lagrangian scheme so that it becomes *inherently* conserving. We will look at both approaches in this paper, and discuss the activities planned in this field at ECMWF.

3. *A posteriori* fixes

The idea behind the *a posteriori* fix is simply to compute the gain or loss of the quantity X, integrated over the domain, at the end of the "dynamical" part of each timestep, and then to adjust the new field of X to restore conservation. The question then arises as to how and where the adjustments should be made. The simplest procedure would be to subtract or add the same amount everywhere in the domain, but a more satisfactory approach is to try to make the adjustments in those regions where the preliminary solution is

most likely to be in error, typically in regions of large gradient. *Priestley* (1993) described such a scheme in which the local error in the preliminary solution was estimated in terms of the difference between a high-order (cubic) and low-order (linear) interpolated value at the departure point of each semi-Lagrangian trajectory. A similar but somewhat more sophisticated scheme has recently been proposed by *Bermejo and Conde* (2002). Both these schemes are in fact extensions of the quasi-monotone semi-Lagrangian algorithm of *Bermejo and Staniforth* (1992).

4. Cell-integrated schemes

The *a posteriori* fixes described above are clearly somewhat arbitrary, and a more pleasing solution would be to design an *inherently* conserving semi-Lagrangian scheme. Cell-integrated schemes form such a class. The underlying idea is the following: instead of finding the departure point corresponding to each arrival gridpoint, we find the departure points corresponding to the corners of the cell surrounding each arrival gridpoint, thus defining a "departure cell". The conserved quantity X can then be integrated over the departure cell (using some assumed spatial distribution), transported to the arrival cell and "remapped" in such a way that the integral is conserved. An early example of such a scheme was that suggested by *Rančić* (1992). *Laprise and Plante* (1995) proposed a variant in which either upstream or downstream trajectories could be used. For a different but related approach, see *Lin and Rood* (1996); also *Leonard et al.* (1996). *Nair and Machenhauer* (2002) designed a cell-integrated semi-Lagrangian scheme for advection on the sphere. The main drawback of cell-integrated schemes is that their complexity (and computational expense) increases rapidly as we move up from one dimension, through two dimensions to three-dimensional problems.

5. Cascade interpolation to the rescue!

Cascade interpolation was first proposed by *Purser and Leslie* (1991), not with conservation in mind but rather to improve the computational efficiency of semi-Lagrangian schemes. In two dimensions, assuming Cartesian geometry (x,y) with a rectangular mesh, the procedure is as follows:

1. First find the departure points as usual, then use them to construct a "Lagrangian" mesh, denoted (X,Y);

2. Find the points at which the Lagrangian Y-lines intersect the Eulerian x-lines;

3. Interpolate (one-dimensionally) along the Eulerian x-lines for the values at these intersection points;

4. Finally, interpolate (one-dimensionally) along the Lagrangian Y-lines for the values at the departure points.

The algorithm can be extended to three dimensions. In general, the multi-dimensional interpolation is replaced by a *cascade* of one-dimensional interpolations, and for high-order interpolation in particular this brings about a significant decrease in the computational burden. Moreover it becomes simpler to incorporate the ideas from cell-integrated schemes since they can be applied in one dimension at a time to produce a *conserving* cascade interpolation scheme. The first such scheme was proposed by *Leslie and Purser* (1995).

Nair et al. (1999a) suggested a slightly simplified and more efficient version of cascade interpolation in Cartesian geometry, which they subsequently extended to the sphere (*Nair et al.*, 1999b). Recently, *Zerroukat et al.* (2002) incorporated conservation properties into this version of cascade interpolation in Cartesian geometry, and finally (*Zerroukat et al.*, 2004) in spherical geometry. Thus the ingredients are in place to

construct global semi-Lagrangian numerical weather prediction and climate models which *inherently* conserve mass and advected tracers.

6. Some problems

Cascade interpolation on the sphere is tricky in the vicinity of the poles, since the Lagrangian and Eulerian meshes are not guaranteed to intersect in the same straightforward way as in Cartesian geometry. Thus, the schemes of *Nair et al.* (1999b), *Nair and Machenhauer* (2002) and *Zerroukat et al.* (2004) all involve a certain amount of "engineering" near the poles.

In the case of the ECMWF model, the use of a "reduced" Gaussian grid (*Hortal and Simmons*, 1991) poses further difficulties since cascade interpolation relies on a "tensor product" grid and it is not obvious how to extend it to the reduced grid.

Conservative cascade interpolation on the sphere requires numerical integration of quantities around lines of latitude and longitude, and it thus less "local" than simple interpolation schemes. Implementation on distributed memory machines will therefore demand some additional thought about communication between processors.

7. ECMWF plans

At ECMWF we plan in the near future to explore the incorporation of conservation properties into the model's semi-Lagrangian scheme. It should be instructive to examine diagnostically the lack of conservation in the present scheme, and to investigate whether the application of an *a posteriori* fix makes a significant difference to any aspect of the predicted fields.

In the longer term, it is hoped to incorporate a conservative cascade interpolation scheme into the semi-Lagrangian time-integration algorithm. If the reduced grid problem proves insurmountable, then it would always be possible to revert to a full grid for applications which really demand that conservation properties be respected.

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