Recent Advances in Global Nonhydrostatic Modeling at NCEP

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1. Introduction

Regional numerical weather prediction (NWP) models have reached the limits of validity of the hydrostatic approximation, and the global ones are rapidly approaching these limits. Vast experience with nonhydrostatic models has been accumulated in simulating convective clouds and storms, but this experience may not be directly or entirely applicable in NWP applications since NWP deals with phenomena on a much wider range of temporal and spatial scales. In response to this situation, over the last decade or so, nonhydrostatic models specifically intended for weather forecasting have been developed and implemented (e.g. Davies et al., 2005; Doms and Schaettler, 1997; Janjic et al., 2001; Janjic, 2003; Room et al., 2006; Saito et al., 2007; Skamarock and Klemp, 2008; Steppeler et al. 2003; Yeh et al., 2002).

Concerning the criteria that a successful nonhydrostatic NWP model should satisfy, there are several, rather obvious choices. Apparently, the accuracy of the nonhydrostatic model must not be inferior to that of mature hydrostatic models running at the same resolution. Moreover, particularly having in mind the uncertainties concerning the benefits that can be expected from nonhydrostatic dynamics at transitional, single digit resolutions in kilometers, the nonhydrostatic model should be sufficiently computationally efficient. Finally, the model dynamics should be capable of reproducing strongly nonhydrostatic flows at very high resolutions. Although such resolutions are beyond the resolutions that could be used in NWP in the near future, this condition must be satisfied in order to demonstrate that the model is indeed nonhydrostatic.

Having in mind these considerations, a novel approach (Janjic et al., 2001; Janjic, 2003) has been applied in the NCEP regional Nonhydrostatic *Mesoscale* Model (NMM) that was developed within the Weather Research and Forecasting (WRF) initiative, and in the new unified Nonhydrostatic *Multiscale* Model on the Arakawa B grid (NMMB) (Janjic, 2005; Janjic and Black, 2007) that is being developed at NCEP within the NOAA Environmental Modeling System (NEMS). The NMMB is designed for a broad range of spatial and temporal scales so that it can be used for a variety of applications from LES studies to weather forecasting and climate simulations on regional and global scales.

In the NMM and NMMB models the hydrostatic approximation is relaxed in hydrostatic formulations based on modeling principles proven in practice. These principles were applied in several generations of models preceding the NMM and NMMB (e.g., Janjic 1977, 1979, 1984), and have been thoroughly tested in NWP and regional climate applications, although the specific numerical schemes employed have evolved significantly over time, and over about two orders of magnitude in resolution. By relaxing the hydrostatic approximation, the applicability of the model formulation is extended to nonhydrostatic motions, and at the same time, the favorable features of the hydrostatic formulation are preserved. In other words, following an evolutionary approach, the nonhydrostatic NWP models were built on NWP experience. The nonhydrostatic effects are introduced in the form of an add-on nonhydrostatic module that can be turned on or off. Following the prevalent NWP practice, the NMM and NMMB were formulated using a vertical coordinate based on mass (or hydrostatic pressure) (Janjic et al., 2001; Janjic, 2003).

2. Nonhydrostatic model equations

Let s denote a generalized mass based terrain following vertical coordinate that varies from 0 at the model top to 1 at the surface (Simmons and Burridge, 1981). Let π be the hydrostatic pressure, and let π_S and π_T be the hydrostatic pressures at the surface and at the top of the model atmosphere, respectively. Then, the difference in hydrostatic pressure between the base and the top of the model column is $\mu = \pi_S - \pi_T$. Here, π_T is a nonnegative constant, whereas π_S is a function of time and horizontal position.

The hypsometric equation

$$\frac{\partial \Phi}{\partial \pi} = -\alpha \tag{2.1}$$

relates the geopotential Φ to the hydrostatic pressure. Assuming that the atmosphere is dry, the specific volume is related to the temperature T and pressure p by the ideal gas law $\alpha = RT/p$, R being the gas constant. Note that the ideal gas law does not involve the hydrostatic pressure but rather the actual pressure. Using the ideal gas law, from (2.1),

$$\frac{\partial \Phi}{\partial s} = -\frac{RT}{p}\frac{\partial \pi}{\partial s}. \tag{2.2}$$

Upon integration of (2.2) from the surface, where the geopotential is denoted by Φ_S, to an arbitrary level s,

$$\Phi = \Phi_s + \int_s^1 \frac{RT}{p}\frac{\partial \pi}{\partial s'} ds'. \tag{2.3}$$

Using (2.1), the third equation of motion may be written as

$$\frac{dw}{dt} = g\left(\frac{\partial p}{\partial \pi} - 1\right). \tag{2.4}$$

Defining the ratio of the vertical acceleration and gravity g,

$$\varepsilon \equiv \frac{1}{g}\frac{dw}{dt}, \tag{2.5}$$

(2.4) may be rewritten as

$$\frac{\partial p}{\partial \pi} = 1 + \varepsilon, \tag{2.6}$$

which defines the relationship between the hydrostatic and the nonhydrostatic pressures. Integrating (2.6) with respect to π, one obtains the nonhydrostatic pressure at an arbitrary hydrostatic pressure, or on a coordinate surface s,

$$p = \int_{\pi_T}^{\pi} \frac{\partial p}{\partial \pi'} d\pi' = \int_{0}^{s} (1+\varepsilon) \frac{\partial \pi'}{\partial s'} ds' . \tag{2.7}$$

As can be seen from (2.6) and (2.7), should ε vanish, the pressure and the hydrostatic pressure become equivalent.

In the hydrostatic s coordinate system, the time derivative of a fluid property q following the motion of an air parcel may be written as

$$\frac{dq}{dt} = \left(\frac{\partial q}{\partial t}\right)_s + \mathbf{v} \cdot \nabla_s q + \left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial q}{\partial \pi} . \tag{2.8}$$

Here, $\dot{s}$ is the vertical velocity and the subscripts indicate the variable that is kept constant while the differentiation is performed.

The nonhydrostatic continuity equation takes the form

$$w = \frac{1}{g}\left[\left(\frac{\partial \Phi}{\partial t}\right)_s + \mathbf{v} \cdot \nabla_s \Phi + \left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial \Phi}{\partial \pi}\right], \tag{2.9}$$

i.e., reduces to the definition of the vertical velocity w. The familiar hydrostatic mass continuity equation

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \pi}{\partial s}\right)\right]_s + \nabla_s \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s}\right) + \frac{\partial}{\partial s}\left(\dot{s}\frac{\partial \pi}{\partial s}\right) = 0 \tag{2.10}$$

also follows from the nonhydrostatic continuity equation.

Using the material surface boundary conditions $\dot{s} \equiv ds/dt$ at $s = 0$ and $s = 0$, one may obtain two equations from (2.10). The first one gives the tendency of the hydrostatic surface pressure

$$\frac{\partial \mu}{\partial t} = -\int_0^1 \nabla_{s'} \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right) ds' , \tag{2.11}$$

and the second one is used to calculate the vertical velocity term

$$\left(\dot{s}\frac{\partial \pi}{\partial s}\right)_s = -\left(\frac{\partial \pi}{\partial t}\right)_s - \int_0^s \nabla_{s'} \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right) ds' . \tag{2.12}$$

Using the relations (2.1) and (2.6), in the case of a nonhydrostatic atmosphere one obtains

$$-\frac{1}{\rho}\nabla_z p \equiv -(1+\varepsilon)\nabla_s \Phi - \alpha \nabla_s p . \tag{2.13}$$

Using (2.13), the inviscid nonhydrostatic equation for the horizontal part of the wind takes the form

$$\frac{d\mathbf{v}}{dt} = -(1+\varepsilon)\nabla_s \Phi - \alpha \nabla_s p + f\mathbf{k} \times \mathbf{v} . \tag{2.14}$$

Again, for vanishing ε, (2.14) reduces to the form used in hydrostatic models.

The first law of thermodynamics for adiabatic processes has the form

$$c_p \frac{dT}{dt} = \alpha \frac{dp}{dt} \tag{2.15}$$

in which c_p is the specific heat at constant pressure. In hydrostatic models, the derivative dp/dt is replaced by the derivative of hydrostatic pressure $d\pi/dt$, often denoted by the Greek letter omega. For this reason, the right hand side of the equation is frequently referred to as the "omega–alpha" term. The derivative of pressure can be separated into a component ω_1 which reduces to the hydrostatic expression when ε vanishes, and a component ω_2 which vanishes with vanishing ε. Note that, generally, $p = p(x, y, \pi, t)$. Then

$$\frac{\partial p}{\partial t} = \left(\frac{\partial p}{\partial \pi}\right)_t \frac{\partial \pi}{\partial t} + \left(\frac{\partial p}{\partial t}\right)_\pi = (1+\varepsilon)\frac{\partial \pi}{\partial t} + \left(\frac{\partial p}{\partial t}\right)_\pi , \tag{2.16}$$

where the subscripts indicate the variable that is kept constant while the differentiation is performed. In addition, as can be seen from (2.6),

$$\left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial p}{\partial \pi} = (1+\varepsilon)\left(\dot{s}\frac{\partial \pi}{\partial s}\right). \tag{2.17}$$

Thus, dp/dt is written in the form

$$\frac{dp}{dt} = \omega_1 + \omega_2 \tag{2.18}$$

where

$$\omega_1 \equiv (1+\varepsilon)\frac{\partial \pi}{\partial t} + \mathbf{v}\cdot\nabla_s p + (1+\varepsilon)\left(\dot{s}\frac{\partial \pi}{\partial s}\right), \tag{2.19}$$

or taking into account (2.12),

$$\omega_1 = \mathbf{v}\cdot\nabla_s p - (1+\varepsilon)\int_0^s \nabla_{s'}\cdot\left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right)ds' . \tag{2.20}$$

Note that the contribution of the second term of the pressure gradient force in (2.13) to the kinetic energy generation is compensated by the contribution of the horizontal advection of pressure in (2.19). The second part of ω is defined by

$$\omega_2 \equiv \frac{\partial p}{\partial t} - (1+\varepsilon)\frac{\partial \pi}{\partial t} . \tag{2.21}$$

Note that the term (2.21) vanishes for vanishing ε.

In view of the separation of omega into two parts, the thermodynamic equation is separated into two parts as well,

$$\left(\frac{\partial T}{\partial t}\right)_1 = -\mathbf{v}\cdot\nabla_s T - \left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial T}{\partial \pi} + \frac{1}{c_p}(\alpha\omega_1) \tag{2.22}$$

and

$$\left(\frac{\partial T}{\partial t}\right)_2 = \frac{1}{c_p}(\alpha\omega_2) . \tag{2.23}$$

With the aid of (2.20), (2.22) may be rewritten as

$$\left(\frac{\partial T}{\partial t}\right)_1 = -\mathbf{v}\cdot\nabla_s T - \left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial T}{\partial \pi} + \frac{\alpha}{c_p}\left[\mathbf{v}\cdot\nabla_s p - (1+\varepsilon)\int_0^s \nabla_{s'}\cdot\left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right)ds'\right]. \tag{2.24}$$

Again, when ε vanishes, (2.22) and (2.24) take the form used in hydrostatic models, and the equation for the second part (2.23) takes the trivial form $(\partial T/\partial t)_2 = 0$.

The nonhydrostatic system of equations is closed by applying the operator (2.8) to the continuity equation (2.9) in order to obtain the vertical acceleration dw/dt. Then, from (2.5),

$$\varepsilon = \frac{1}{g}\frac{dw}{dt} = \frac{1}{g}\left[\left(\frac{\partial w}{\partial t}\right)_s + \mathbf{v}\cdot\nabla_s w + \left(\dot{s}\frac{\partial \pi}{\partial s}\right)\frac{\partial w}{\partial \pi}\right]. \tag{2.25}$$

The parameter ε is the central point of the extended, nonhydrostatic dynamics. Assume for a moment that ε is zero. Then, Eqs. (2.2), (2.10), (2.14) and (2.24), together with the gas law, represent the set of equations describing the hydrostatic, inviscid, adiabatic atmosphere. However, the presence of nonzero ε in (2.6), (2.14) and (2.24) demonstrates in a transparent way where, how, and to what extent relaxing the hydrostatic approximation affects the familiar hydrostatic equations. Note that the system of equations developed above bears a close relation to the system discussed by Laprise (1992).

On the synoptic scales, ε is small and approaches the computer round–off error. However, in case of vigorous convective storms, or strong vertical accelerations in flows over steep obstacles, the vertical velocity can reach the order of $10ms^{-1}$ over the period of the order of $1000s$. This yields an estimate of the vertical acceleration of the order of $10^{-2}ms^{-2}$, and consequently, ε of the order of 10^{-3}. As can be seen from (2.6), for this value of ε the nonhydrostatic deviation of pressure can reach $100Pa$. Bearing in mind that the typical synoptic scale horizontal pressure gradient is of the order of $100Pa$ over $100km$, this suggests that significant local nonhydrostatic pressure gradients and associated circulations may develop on small scales. Nevertheless, ε remains much smaller than unity in atmospheric flows, and therefore, the nonhydrostatic effects in (2.6), (2.14) and (2.24) are of a higher order magnitude. An important consequence of this situation for the discretization is that high accuracy of computation of ε does not appear to be of paramount importance, since the computational errors are of even higher order than ε.

As can be seen from (2.2), the geometric height z is uniquely defined by the hydrostatic and nonhydrostatic pressures π and p, and temperature T. Thus, if these three variables are known, the vertical velocity w can be computed using the definition (2.8), or the nonhydrostatic continuity equation (2.9). Hence, w (and consequently ε) cannot be considered as an independent prognostic variable. Nevertheless, for internal consistency, the vertical velocity w must also satisfy the prognostic vertical equation of motion (2.4). In the next section it will be shown how this can be done.

3. Time stepping

The NMM and NMMB time stepping scheme is presented in this section in order to illustrate how the extra terms appearing in the nonhydrostatic system are treated. The economical forward–backward scheme (Ames, 1969; Gadd, 1974; Janjic and Wiin-Nielsen, 1977; Janjic 1979) is used for the adjustment terms. Modified Adams-Bashforth scheme is used for the horizontal advection terms and the Coriolis force. Although the Adams-Bashforth scheme is slightly unstable, the instability is very weak so that the scheme can be safely used in practice. Nevertheless, for conceptual reasons, in the scheme used in the NMM and NMMB the instability is removed by slight off-centering. After the off-centering, the modified scheme becomes weakly dissipative (Janjic et al., 2010). The Crank-Nicholson scheme is used for the vertical advection.

The superscripts n and n+1 will be used to denote the time levels for all variables with the exception of the vertical velocity w which is defined at intermediate time levels indicated by superscripts $n+1/2$ or $n-1/2$. The superscript $n+1/2$ will be used also in the advection and Coriolis force terms in order to indicate that centered in time schemes are used. Because the nonhydrostatic equations have been separated into two components, the subscript 1 will be used to indicate that a variable has been advanced in time only by the first component equation. For example, the solution of (2.24) starting from the time level n will be denoted by the subscript 1, since (2.23) remains to be solved before reaching the time level $n+1$.

The vertical velocity term in the hydrostatic s coordinate is computed integrating (2.10)

$$\left(\dot{s}\frac{\partial \pi}{\partial s}\right)_s^n = -\left(\frac{\partial \pi}{\partial t}\right)_s^n - \int_0^s \nabla_{s'} \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right)^n ds' , \tag{3.1}$$

and the surface pressure tendency equation is

$$\mu^{n+1} = \mu^n - \Delta t \int_0^1 \nabla_{s'} \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right)^n ds' . \tag{3.2}$$

The first component of the nonhydrostatic pressure is computed using

$$p_1 = p^n + (1+\varepsilon^n)\Delta t \left(\frac{\partial \pi}{\partial t}\right)^n . \tag{3.3}$$

Then

$$\omega_1 = \mathbf{v}^n \cdot \nabla_s p^n - (1+\varepsilon^n)\int_0^s \nabla_{s'} \cdot \left(\mathbf{v}\frac{\partial \pi}{\partial s'}\right)^n ds' , \tag{3.4}$$

and the first component of the thermodynamic equation is

$$T_1 = T^n + \frac{\Delta t}{c_p}\frac{RT^n}{p^n}\omega_1 - \Delta t\left[\mathbf{v}^n \cdot \nabla_s T^{n+1/2} - \left(\dot{s}\frac{\partial \pi}{\partial s}\right)^n \frac{\partial T^{n+1/2}}{\partial \pi^n}\right] . \tag{3.5}$$

The superscript $n+1/2$ in the advection terms indicates symbolically that centered schemes are used.

The second component of the thermodynamic equation is

$$T^{n+1} - T_1 = \frac{1}{c_p}\frac{RT_1}{p_1}(p^{n+1} - p_1) \tag{3.6}$$

The hypsometric equation yields the geopotential associated with the first component solutions for temperature and pressure,

$$\Phi_1 = \Phi_S + \int_s^1 \frac{RT_1}{p_1}\left(\frac{\partial \pi}{\partial s'}\right)^{n+1} ds' \tag{3.7}$$

and the second component equation yields

$$\Phi^{n+1} = \Phi_S + \int_s^1 \frac{RT^{n+1}}{p^{n+1}} \left(\frac{\partial \pi}{\partial s'} \right)^{n+1} ds' \,. \tag{3.8}$$

The value of vertical velocity w associated with the first component solutions is obtained from

$$g\, w_1 = \frac{\Phi_1 - \Phi^n}{\Delta t} + \mathbf{v}^n \cdot \nabla_s \Phi_1 + \left(\dot{s} \frac{\partial \pi}{\partial s} \right)^n \frac{\partial \Phi_1}{\partial \pi^n} \,. \tag{3.9}$$

Note that Φ_1 is an intermediate value of geopotential between Φ^n and Φ^{n+1}, i.e.,

$$\Phi^{n+1} - \Phi_1 \le O(\Delta t) \,. \tag{3.10}$$

Therefore, using Φ_1 in the advection terms of (3.9) in order to compute w is a consistent numerical approximation. On the other hand, neglecting the contribution $(\Phi^{n+1} - \Phi_1)/\Delta t$ would be wrong in view of (3.10). Thus,

$$w^{n+1/2} - w_1 = \frac{\Phi^{n+1} - \Phi_1}{g\Delta t} \,, \tag{3.11}$$

which must also satisfy the 3rd equation of motion,

$$w^{n+1/2} - g\Delta t \frac{\partial p^{n+1}}{\partial \pi^{n+1}} = w_1 - g\Delta t (1 + \varepsilon_1) \,. \tag{3.12}$$

The value of ε associated with the first component solutions is obtained from

$$g\, \varepsilon_1 = \frac{w_1 - w^{n-1/2}}{\Delta t} + \mathbf{v}^n \cdot \nabla_\sigma w_1 + \left(\dot{s} \frac{\partial \pi}{\partial s} \right)^n \frac{\partial w_1}{\partial \pi^n} \,, \tag{3.13}$$

and the second component from

$$\varepsilon^{n+1} = \frac{\partial p^{n+1}}{\partial \pi^{n+1}} - 1 \,. \tag{3.14}$$

Upon solution of the preceding equations for thermodynamic variables, the pressure gradient force at the time level n+1 can be computed, and the horizontal equation of motion can be used to advance the wind components in time and thus complete the time step. The superscript $n + 1/2$ in the advection and Coriolis terms again indicates that centered schemes are used

$$\begin{aligned} \mathbf{v}^{n+1} = \mathbf{v}^n &- \Delta t \left[(1 + \varepsilon^{n+1}) \nabla_s \Phi^{n+1} - \alpha^{n+1} \nabla_s p^{n+1} + f \mathbf{k} \times \mathbf{v}^{n+1/2} \right] \\ &- \Delta t \left[\mathbf{v}^n \cdot \nabla_\sigma \mathbf{v}^{n+1/2} + \left(\dot{s} \frac{\partial \pi}{\partial s} \right)^n \frac{\partial \mathbf{v}^{n+1/2}}{\partial \pi^n} \right] \end{aligned} \,. \tag{3.16}$$

Here, the specific volume α^{n+1} is

$$\alpha^{n+1} = \frac{RT^{n+1}}{p^{n+1}} \tag{3.17}$$

4. Solution of the coupled equations

Eqs. (3.6), (3.8), (3.11) and (3.12) are coupled equations. Their solution will be sought by eliminating all unknowns except p^{n+1}, solving the resulting equation, and then back–substituting to obtain

T^{n+1}, Φ^{n+1}, $w^{n+1/2}$ and ε^{n+1}. Namely, (3.7) and (3.8), together with (3.11) and (3.12), can be combined to give

$$R\int_s^1 \left(\frac{T^{n+1}}{p^{n+1}} - \frac{T_1}{p_1}\right)\left(\frac{\partial \pi}{\partial s'}\right)^{n+1} ds' = (g\Delta t)^2 \left[\frac{1}{\left(\frac{\partial \pi}{\partial s'}\right)^{n+1}} \frac{\partial p^{n+1}}{\partial s} - (1+\varepsilon_1)\right]. \tag{4.1}$$

Using (3.6) to eliminate T^{n+1}, (4.1) may be rewritten as

$$R(1-\kappa)\int_s^1 T_1\left(\frac{1}{p^{n+1}} - \frac{1}{p_1}\right)\left(\frac{\partial \pi}{\partial s'}\right)^{n+1} ds' = (g\Delta t)^2 \left[\frac{1}{\left(\frac{\partial \pi}{\partial s'}\right)^{n+1}} \frac{\partial p^{n+1}}{\partial s} - (1+\varepsilon_1)\right] \tag{4.2}$$

where $\kappa \equiv R/c_p$. Define a pressure p^* that satisfies the equation

$$\frac{1}{\left(\frac{\partial \pi}{\partial s'}\right)^{n+1}} \frac{\partial p^*}{\partial s} \equiv (1+\varepsilon_1) \tag{4.3}$$

subject to the boundary condition $p^* = \pi_T$ at $s = 0$. Upon inserting (4.3) into (4.2), one obtains

$$\frac{R(1-\kappa)}{g^2}\int_s^1 T_1\left(\frac{1}{p^{n+1}} - \frac{1}{p_1}\right)\left(\frac{\partial \pi}{\partial s'}\right)^{n+1} ds' = \Delta t^2 \frac{1}{\left(\frac{\partial \pi}{\partial s'}\right)^{n+1}} \frac{\partial (p^{n+1} - p^*)}{\partial s}. \tag{4.4}$$

As pointed out in Janjic et al. (2001), after some manipulation, (4.4) can be solved iteratively. However, as will be shown here, it can be solved directly as well. Note that

$$p^{n+1} - p^* \le O(\Delta t), \tag{4.5}$$

so that, from (4.4),

$$\frac{R(1-\kappa)}{g^2}\int_s^1 T_1\left(\frac{1}{p^{n+1}} - \frac{1}{p_1}\right)\left(\frac{\partial \pi}{\partial s'}\right)^{n+1} ds' \le O(\Delta t^3), \tag{4.6}$$

which illustrates how subtle is the difference between p_1 and p^{n+1}. In order to visualize more clearly the procedure used to solve (4.4) for p^{n+1} directly, it is convenient to consider vertically discretized form of (4.4). Let each of the lm model layers be denoted by index l increasing from top down, and let the corresponding layer interfaces be denoted by half-indices $l-1/2$ and $l+1/2$. In addition, let temperature be defined at mid-layers, and pressure variables at layer interfaces. Then, using the simplest vertical two-point averaging and differencing operators denoted, respectively, by an overbar and symbol Δ, a discrete version of (4.4) can be written as

$$\frac{R(1-\kappa)}{(g\Delta t)^2}\sum_{k=l}^{k=lm} T_{1l}\left(\frac{\overline{p_{1}}^{\pi}_{l} - \overline{p^{n+1}}^{\pi}_{l}}{\overline{p^{n+1}}^{\pi}_{l}\,\overline{p_{1}}^{\pi}_{l}}\right)\Delta\pi_l^{n+1} = \frac{\Delta p_l^{n+1} - \Delta p_l^*}{\Delta \pi_l^{n+1}}. \tag{4.7}$$

Define a pressure variable

$$p_{2l+1/2} = p_{1l+1/2} + const\left(p_{2l-1/2} + \varepsilon_1 \Delta\pi_l - p_{1l+1/2}\right) \tag{4.8}$$

by correcting p_1 using the latest preliminary value of ε_1. Note that when $const = 0$, $p_{2l+1/2} = p_{1l+1/2}$, and when $const = 1$, $p_{2l+1/2} = p^*_{l+1/2}$. Then, taking into account (4.6), (4.7) can be rewritten as

$$\frac{R(1-\kappa)}{(g\Delta t)^2}\sum_{k=l}^{k=lm}\frac{T_{1l}}{\overline{p}_{2l}^2}\left(\overline{p_{1l}}^{\pi} - \overline{p^{n+1}}^{\pi}_{l}\right)\Delta\pi_l^{n+1} = \frac{\Delta p_l^{n+1} - \Delta p_l^*}{\Delta\pi_l^{n+1}}. \tag{4.9}$$

In practice, for historical reason, the correction weight $const$ in (4.8) is 0.35, but noticeable impact on the solution could not be detected when varying this parameter between 0 and 1. Subtracting from (4.9) analogous expression defined on the level $l+1$, one obtains

$$\frac{R(1-\kappa)}{(g\Delta t)^2}\frac{T_{1l}}{\overline{p}_{2l}^2}\left(\overline{p_{1l}}^{\pi} - \overline{p^{n+1}}^{\pi}_{l}\right)\Delta\pi_l^{n+1} = \frac{\Delta p_l^{n+1} - \Delta p_l^*}{\Delta\pi_l^{n+1}} - \frac{\Delta p_{l+1}^{n+1} - \Delta p_{l+1}^*}{\Delta\pi_{l+1}^{n+1}}. \tag{4.10}$$

Replacing averaging and differencing operators applied to p^{n+1} by explicit algebraic expressions, (4.10) can be rewritten as

$$\begin{aligned}&\frac{R(1-\kappa)}{(g\Delta t)^2}\frac{T_{1l}}{\overline{p}_{2l}^2}\left[\overline{p_{1l}}^{\pi} - \frac{1}{2}\left(p_{l-1/2}^{n+1} + p_{l+1/2}^{n+1}\right)\right]\Delta\pi_l^{n+1} = \\ &\frac{p_{l+1/2}^{n+1} - p_{l-1/2}^{n+1} - \Delta p_l^*}{\Delta\pi_l^{n+1}} - \frac{p_{l+3/2}^{n+1} - p_{l+1/2}^{n+1} - \Delta p_{l+1}^*}{\Delta\pi_{l+1}^{n+1}}\end{aligned}. \tag{4.11}$$

Inspection of (4.11) reveals that the unknown p^{n+1} appears at three consecutive layer interfaces, $l-1/2$, $l+1/2$ and $l+3/2$. Thus (4.11) is a tridiagonal system which can be solved with suitably chosen boundary conditions. A solution without the approximation (4.8) can be obtained by iterating (4.11), but that appears pointless in the light of (4.6).

In order to address the problem of specification of boundary conditions for (4.11), consider a horizontally homogenous atmosphere at rest and in hydrostatic equilibrium. Let the equations be linearized around such a basic state. Also, consider only the solutions that preserve horizontal homogeneity. As can be readily verified, the requirement for horizontal homogeneity eliminates all motions that belong to the first part of the time stepping procedure. In other words, the intermediate solutions denoted by subscript 1 will coincide with the initial values denoted by superscript *n*. The only solutions left will be those described by the linearized set of coupled equations leading to (4.4). In particular, from (3.6)

$$(T^{n+1} - T_0) - (T^n - T_0) = \frac{1}{c_p}\frac{RT_0}{\pi_0}\left[(p^{n+1} - \pi_0) - (p^n - \pi_0)\right], \tag{4.12}$$

and after differentiation of (3.8) with respect to s, linearization and rearrangement,

$$g\frac{\partial(z^{n+1} - z_0)}{\partial\pi_0} = -\frac{R(T^{n+1} - T_0)}{\pi_0} + \frac{RT_0}{\pi_0}\frac{(p^{n+1} - \pi_0)}{\pi_0}. \tag{4.13}$$

Here, z is the height and the subscript 0 denotes the basic state variables. From (3.11) and (3.12)

$$w^{n+1/2} = \frac{(z^{n+1} - z_0) - (z^n - z_0)}{\Delta t} \tag{4.14}$$

$$\frac{w^{n+1/2} - w^{n-1/2}}{\Delta t} = g \frac{\partial (p^{n+1} - \pi_0)}{\partial \pi_0} . \tag{4.15}$$

Introducing primes to denote the deviations from the basic state, applying the simplest time differencing operator to (4.12) and using (4.14),

$$g \frac{\partial w^{n+1/2}}{\partial \pi_0} = -\frac{R(T'^{n+1} - T'^{n})}{\pi_0 \Delta t} + \frac{RT_0}{\pi_0} \frac{(p'^{n+1} - p'^{n})}{\pi_0 \Delta t} . \tag{4.16}$$

Using (4.12) to eliminate *T'* in (4.16), and differencing in time the resulting equation, one obtains

$$g \frac{\partial}{\partial \pi_0} \frac{w^{n+1/2} - w^{n-1/2}}{\Delta t} = \frac{c_v}{c_p} \frac{RT_0}{\pi_0^2} \frac{(p'^{n+1} - 2p'^{n} + p'^{n-1})}{\Delta t^2} . \tag{4.17}$$

On the other hand, differentiating (4.15) with respect to π_0,

$$\frac{\partial}{\partial \pi_0} \frac{w^{n+1/2} - w^{n-1/2}}{\Delta t} = g \frac{\partial^2 p'^{n+1}}{\partial \pi_0^2} . \tag{4.18}$$

Thus, combining (4.17) and (4.18), and taking into account that the basic state is hydrostatic,

$$\frac{p'^{n+1} - 2p'^{n} + p'^{n-1}}{\Delta t^2} = \frac{c_p}{c_v} RT_0 \frac{\partial^2 p'^{n+1}}{\partial z_0^2} . \tag{4.19}$$

The equation for vertically propagating sound waves is readily recognized in (4.19), although finite differencing is used instead of differentiation with respect to time on the left–hand side.

Now that the physical nature of the processes involved in the second part of the integration procedure have been revealed, the question of the boundary conditions for (4.4) can be readdressed. It appears natural to keep the upper end of the oscillator described by (4.19) fixed, and the lower end free. Thus, $p = \pi$ is set at $s = 0$, and $\partial(p^{n+1} - p^*)/\partial s = 0$ is set at $s = 1$. Such an upper boundary condition is perfectly justified for vanishing pressure at the top of the atmosphere of the model.

5. NCEP's unified grid point nonhydrostatic multiscale model

The new unified Nonhydrostatic *Multiscale* Model on the Arakawa B grid (NMMB) has been under development at the National Centers for Environmental Prediction (NCEP) within the new NOAA Environmental Modeling System (NEMS) (Janjic, 2005; Janjic and Black, 2007). The model is designed for a broad range of spatial and temporal scales so that it can be applied for a variety of applications from LES studies to weather forecasting and climate simulations on regional and global scales. The NMMB represents the second generation of grid point nonhydrostatic models developed at NCEP. Except for being reformulated for the B grid, the model formulation follows the general modeling philosophy of its predecessor, the NCEP's regional Nonhydrostatic *Mesoscale* Model (WRF NMM). The WRF NMM has been used for various applications at NCEP and elsewhere since early 2000's, and since 2006 it has become the main short-range forecasting regional North American Model (NAM).

The NCEP grid point nonhydrostatic models have been developed building on NWP experience (Janjic et al., 2001; Janjic 2003). As in their hydrostatic predecessors, mass based hydrostatic vertical coordinates have been used. With the mass (hydrostatic pressure) coordinate the nondivergent flow remains on coordinate surfaces. Note that a similar argument applies to adiabatic flows in isentropic coordinates. However, important flow regimes on the meso scales are characterized by weak stability and strong diabatic forcing, which renders the isentropic coordinates less appealing (although still applicable) on these scales. With this choice, the mass, as well as a number of other first order and quadratic quantities can be conserved in the discrete system in a straightforward way.

The nonhydrostatic dynamics were formulated by relaxing the hydrostatic approximation in advanced hydrostatic NWP formulations. In this way, the validity of the hydrostatic model was extended to nonhydrostatic motions, and at the same time the preferable features of the hydrostatic formulation were preserved in the hydrostatic limit. More specifically, as shown in the preceding sections, the system of nonhydrostatic equations in the general mass based vertical coordinate is split into two parts: (a) the part that corresponds to the hydrostatic system, except for higher order corrections due to vertical acceleration, and (b) the system of equations that allows computation of the corrections due to the vertical acceleration. The separation of the nonhydrostatic contributions shows in a transparent way where, how and to what extent the hydrostatic approximation affects the equations. This approach does not require any linearization or additional approximation.

The resulting system of nonhydrostatic equations has only one additional prognostic equation for nonhydrostatic pressure. Given the hydrostatic pressure, nonhydrostatic pressure and temperature, the geopotential is uniquely defined, and vertical velocity and vertical acceleration are computed from geopotential. Of course, as shown in the preceding sections, the prognostic equation for the vertical velocity reduces to a consistent discrete approximation, so that internal consistency of the discrete system is preserved.

The nonhydrostatic dynamics extension is implemented through an add–on nonhydrostatic module. The nonhydrostatic module can be turned on and off depending on resolution in order to eliminate the computational overhead at coarse and transitional resolutions where the impact of nonhydrostatic effects is not detectable. More importantly, this feature allows easy comparison of hydrostatic and nonhydrostatic solutions at various resolutions using identical hydrostatic core.

The "isotropic" quadratic conservative finite-volume horizontal differencing employed in the model conserves a variety of basic and derived dynamical and quadratic quantities and preserves some important properties of differential operators. Among these, the conservation of energy and enstrophy improves the accuracy of the nonlinear dynamics of the model on all scales (Arakawa, 1966; Janjic, 1984). The NMMB uses the regular latitude-longitude grid for the global domain, and a rotated latitude-longitude grid in regional applications. With the Equator of the rotated system running through the middle of the regional integration domain, more uniform grid distances are obtained. In the vertical, the hybrid pressure-sigma coordinate has been chosen as the primary option. "Across the pole" polar boundary conditions are specified in the global limit and the polar filter acting on tendencies selectively slows down the wave components of the basic dynamical variables that would otherwise propagate faster in the zonal direction than the fastest wave propagating in the meridional direction.

In very high resolution tests, a two-dimensional model based on the described principles successfully reproduced the classical two-dimensional nonhydrostatic solutions and thus demonstrated the validity of the concept (Janjic et al., 2001; Janjic, 2004). The high resolution tests using the regional version of the model indicate that the impact of nonhydrostatic dynamics becomes detectable at about 8 km resolution provided almost all dissipative mechanisms in the model are turned off, and noticeable at about 1 km resolution (Janjic et al., 2001; Janjic, 2003, 2004). The extra computational cost of the nonhydrostatic dynamics is on the order of 10% in global applications, or nonexistent if the nonhydrostatic extension is switched off at coarser resolutions. However, the relatively low cost of the nonhydrostatic dynamics allows its application even at transitional resolutions where the benefits due to the nonhydrostatic dynamics are uncertain.

The model has been computationally robust, efficient and reliable in operational applications and pre-operational tests. As indicated by regular runs carried out at NCEP for over a year, the NMMB produces good medium range forecasts, and its computational efficiency compares favorably with other medium range forecasting models.

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