Third-order accurate MPDATA for arbitrary flows

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Highlights of MPDATA

- MPDATA is a sign-preserving second-order accurate scheme for numerical integration of generalized transport equation

  $$\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\boldsymbol{V}\Psi) = GR$$

  based on iterative application of the upwind algorithm
- The basic version with two iterations uses physical "velocity" $\boldsymbol{V}$ in the first iteration and error-correcting antidiffusive velocity $\overline{\boldsymbol{V}}$ in the second
- The form of the antidiffusive velocity is derived by performing truncation analysis of the upwind scheme

Antidiffusive velocity on structured grids

$$\overline{\boldsymbol{V}} = \frac{1}{2}\boldsymbol{\delta x} \odot \uparrow\boldsymbol{V}\uparrow \odot \frac{\nabla\Psi}{\Psi} - \frac{1}{2}\delta t \frac{\boldsymbol{V}}{G}\frac{\nabla \cdot (\boldsymbol{V}\Psi)}{\Psi}$$

where

$$(\uparrow\boldsymbol{a}\uparrow)^I = |a^I|$$

denotes component-wise absolute value of a vector and

$$(\boldsymbol{a} \odot \boldsymbol{b})^I := a^I b^I$$

is the Hadamard product of two vectors,

Third-order accurate MPDATA for time and space dependant flows

The idea

- Find "third-order" antidiffusive velocity $\overline{\overline{\boldsymbol{V}}}$ that compensates MPDATA error for arbitrary flows
- Done for the constant velocity case in L. Margolin, P. K. Smolarkiewicz, SISC, 1998
- Here we consider homogeneous generalized transport equation with G independent of time and arbitrary $\boldsymbol{V}$

$$G\frac{\partial\Psi}{\partial t} + \nabla\cdot(\boldsymbol{V}\Psi) = 0$$

A glimpse at the result

$$\begin{aligned}
\overline{\overline{\boldsymbol{V}}} = &-\frac{\boldsymbol{\delta x}\odot\boldsymbol{\delta x}}{24}\odot\left[4\boldsymbol{V}\odot\frac{1}{\Psi}\nabla\odot\nabla\Psi + 2\frac{\nabla\Psi}{\Psi}\odot\nabla\odot\boldsymbol{V} + \nabla\odot\nabla\odot\boldsymbol{V}\right] \\
&+ \frac{\boldsymbol{\delta x}}{2}\odot\left\uparrow\overline{\boldsymbol{V}}\right\uparrow\odot\frac{\nabla\Psi}{\Psi} + \frac{\delta t}{2}\boldsymbol{\delta x}\odot\uparrow\boldsymbol{V}\uparrow\odot\frac{1}{\Psi}\nabla\left[\frac{1}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right] \\
&+ \frac{\delta t^2}{24}\left\{-\frac{8\boldsymbol{V}}{G\Psi}\nabla\cdot\left[\frac{\boldsymbol{V}}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right] + \frac{\partial^2\boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi}\nabla\cdot\left(\frac{\partial\boldsymbol{V}}{\partial t}\Psi\right) - \frac{2}{G\Psi}\frac{\partial\boldsymbol{V}}{\partial t}\nabla\cdot(\boldsymbol{V}\Psi)\right\}
\end{aligned}$$

Salient points of the derivation

- Starting point is modified equation analysis of the basic MPDATA scheme - different approach than for the constant velocity case
- Taylor expansion results in an equation of the form

$$\frac{\partial \Psi}{\partial t} + \frac{1}{G}\nabla \cdot (\boldsymbol{V}\Psi) = \frac{1}{G}\nabla \cdot (\boldsymbol{T}_0) - \frac{\delta t}{2}\frac{\partial^2 \Psi}{\partial t^2} - \frac{\delta t^2}{6}\frac{\partial^3 \Psi}{\partial t^3}$$

- Following Lax-Wendroff/Cauchy-Kovalevskaya procedure time derivatives are expressed in terms of spatial derivatives

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{G}\nabla \cdot (\boldsymbol{T}_1) \qquad \frac{\partial^3 \Psi}{\partial t^3} = \frac{1}{G}\nabla \cdot (\boldsymbol{T}_2)$$

- Combination of the above leads to

$$\frac{\partial \Psi}{\partial t} + \frac{1}{G}\nabla \cdot (\boldsymbol{V}\Psi) = \frac{1}{G}\nabla \cdot \left(\boldsymbol{T}_0 - \frac{\delta t}{2}\boldsymbol{T}_1 - \frac{\delta t^2}{6}\boldsymbol{T}_2\right) := \frac{1}{G}\nabla \cdot \left(\overline{\overline{\boldsymbol{V}}}\Psi\right)$$

  where the last equation defines third-order antidiffusive velocity $\overline{\overline{\boldsymbol{V}}}$
- Result depends on details of MPDATA implementation (velocity interpolation and extrapolation, antidiffusive velocities time level, ...)
- Finding $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, and $\boldsymbol{T}_2$ is a tedious task suited for the use of computer algebra systems (CAS) e.g. SageMath

Interlude - hidden complexity and perils of CAS

First component of third-order antidiffusive velocity

$$
\begin{aligned}
&+\left[-\frac{dt\,dxU^2G_x}{2\,G^2}+\frac{dt^2U^3G_x}{3\,G^3}+\frac{dt^2U^2VG_y}{3\,G^3}+\frac{dt^2U^2WG_z}{3\,G^3}-\frac{1}{12}\,dx^2U_x+\frac{3\,dt\,dxUU_x}{4\,G}-\frac{dt^2U^2U_x}{G^2}-\frac{dt^2UVU_y}{3\,G^2}-\frac{dt^2UWU_z}{3\,G^2}+\frac{dt\,dxUV_y}{4\,G}\right.\\
&\left.-\frac{2\,dt^2U^2V_y}{3\,G^2}+\frac{dt\,dxUW_z}{4\,G}-\frac{2\,dt^2U^2W_z}{3\,G^2}\right]\frac{\psi_x}{\psi}+\left[-\frac{1}{6}\,dx^2U+\frac{dt\,dxU^2}{2\,G}-\frac{dt^2U^3}{3\,G^2}\right]\frac{\psi_{xx}}{\psi}+\left[-\frac{dt\,dxUVG_x}{2\,G^2}+\frac{dt^2U^2VG_x}{3\,G^3}+\frac{dt^2UV^2G_y}{3\,G^3}+\frac{dt^2UVWG_z}{3\,G^3}\right.\\
&\left.-\frac{dt^2VU_t}{12\,G}-\frac{2\,dt^2UVU_x}{3\,G^2}+\frac{dt^2UV_t}{12\,G}+\frac{dt\,dxUV_x}{2\,G}-\frac{dt^2U^2V_x}{3\,G^2}-\frac{dt^2UVV_y}{G^2}-\frac{dt^2UWV_z}{3\,G^2}-\frac{2\,dt^2UVW_z}{3\,G^2}\right]\frac{\psi_y}{\psi}+\left[\frac{dt\,dxUV}{2\,G}-\frac{2\,dt^2U^2V}{3\,G^2}\right]\frac{\psi_{xy}}{\psi}\\
&+\left[-\frac{dt^2UV^2}{3\,G^2}\right]\frac{\psi_{yy}}{\psi}+\left[-\frac{dt\,dxUWG_x}{2\,G^2}+\frac{dt^2U^2WG_x}{3\,G^3}+\frac{dt^2UVWG_y}{3\,G^3}+\frac{dt^2UW^2G_z}{3\,G^3}-\frac{dt^2WU_t}{12\,G}-\frac{2\,dt^2UWU_x}{3\,G^2}-\frac{2\,dt^2UWV_y}{3\,G^2}+\frac{dt^2UW_t}{12\,G}+\frac{dt\,dxUW_x}{2\,G}\right.\\
&\left.-\frac{dt^2U^2W_x}{3\,G^2}-\frac{dt^2UVW_y}{3\,G^2}-\frac{dt^2UWW_z}{G^2}\right]\frac{\psi_z}{\psi}+\left[\frac{dt\,dxUW}{2\,G}-\frac{2\,dt^2U^2W}{3\,G^2}\right]\frac{\psi_{xz}}{\psi}+\left[-\frac{2\,dt^2UVW}{3\,G^2}\right]\frac{\psi_{yz}}{\psi}+\left[-\frac{dt^2UW^2}{3\,G^2}\right]\frac{\psi_{zz}}{\psi}+\frac{1}{24}\,dt^2U_{tt}+\frac{dt^2UU_{tx}}{12\,G}\\
&-\frac{dt\,dxUG_xU_x}{2\,G^2}+\frac{dt^2U^2G_xU_x}{3\,G^3}+\frac{dt^2UVG_yU_x}{3\,G^3}+\frac{dt^2UWG_zU_x}{3\,G^3}-\frac{dt^2U_tU_x}{12\,G}-\frac{dt^2UU_x^2}{3\,G^2}-\frac{1}{24}\,dx^2U_{xx}+\frac{dt\,dxUU_{xx}}{2\,G}-\frac{dt^2U^2U_{xx}}{3\,G^2}-\frac{dt^2UVU_{xy}}{3\,G^2}\\
&-\frac{dt^2UWU_{xz}}{3\,G^2}+\frac{dt^2UV_{ty}}{12\,G}+\frac{dt\,dxUV_{xy}}{2\,G}-\frac{dt^2U^2V_{xy}}{3\,G^2}-\frac{dt\,dxUG_xV_y}{2\,G^2}+\frac{dt^2U^2G_xV_y}{3\,G^3}+\frac{dt^2UVG_yV_y}{3\,G^3}+\frac{dt^2UWG_zV_y}{3\,G^3}-\frac{dt^2U_tV_y}{12\,G}-\frac{2\,dt^2UU_xV_y}{3\,G^2}\\
&-\frac{dt^2UV_y^2}{3\,G^2}-\frac{dt^2UVV_{yy}}{3\,G^2}-\frac{dt^2UWV_{yz}}{3\,G^2}+\frac{dt^2UW_{tz}}{12\,G}+\frac{dt\,dxUW_{xz}}{2\,G}-\frac{dt^2U^2W_{xz}}{3\,G^2}-\frac{dt^2UVW_{yz}}{3\,G^2}-\frac{dt\,dxUG_xW_z}{2\,G^2}+\frac{dt^2U^2G_xW_z}{3\,G^3}+\frac{dt^2UVG_yW_z}{3\,G^3}\\
&+\frac{dt^2UWG_zW_z}{3\,G^3}-\frac{dt^2U_tW_z}{12\,G}-\frac{2\,dt^2UU_xW_z}{3\,G^2}-\frac{2\,dt^2UV_yW_z}{3\,G^2}-\frac{dt^2UW_z^2}{3\,G^2}-\frac{dt^2UWW_{zz}}{3\,G^2}+\frac{dx^2U\psi_x^2}{4\,\psi^2}-\frac{dt\,dxU^2\psi_x^2}{4\,G\psi^2}-\frac{dt\,dxUV\psi_x\psi_y}{4\,G\psi^2}-\frac{dt\,dxUW\psi_x\psi_z}{4\,G\psi^2}
\end{aligned}
$$

Implementation and stability considerations

Implementation

- The differential expressions or interpolated variables that enter $\overline{\overline{\boldsymbol{V}}}$, for example $\frac{\partial^2 \boldsymbol{V}}{\partial t^2}$ or $\frac{\nabla \Psi}{\Psi}$, have to be known only to the first order
- The entire scheme can be done in two upwind passes, in the second pass $\widetilde{\boldsymbol{V}} = \overline{\boldsymbol{V}}(\boldsymbol{V}) + \overline{\overline{\boldsymbol{V}}}(\overline{\boldsymbol{V}}, \boldsymbol{V})$

Stability

- Upwind passes will be stable if the Courant numbers associated with physical and antidiffusive velocities are properly bounded
- Second-order antidiffusive velocity Courant number is bounded by the physical velocity Courant number given suitable timestep restriction, the same has not been proven for the third-order antidiffusive velocity
- Formally the third-order antidiffusive velocity is $O(\delta t, \boldsymbol{\delta x})$ correction to the second-order antidiffusive velocity and should be small for smooth problems in the high-resolution limit
- From stability point of view it may be beneficial to split second-order and third-order corrections into separate upwind passes
- Some form of antidiffusive velocity limiting may still be needed

A walk-through of the result

Considered term

$$\overline{\overline{\boldsymbol{V}}} = -\frac{\boldsymbol{\delta x} \odot \boldsymbol{\delta x}}{24} \odot \left[4\boldsymbol{V} \odot \frac{1}{\Psi} \nabla \odot \nabla \Psi + 2 \frac{\nabla \Psi}{\Psi} \odot \nabla \odot \boldsymbol{V} + \nabla \odot \nabla \odot \boldsymbol{V} \right]$$
$$+ \frac{\boldsymbol{\delta x}}{2} \odot \left\uparrow \overline{\boldsymbol{V}} \right\uparrow \odot \frac{\nabla \Psi}{\Psi} + \frac{\delta t}{2} \boldsymbol{\delta x} \odot \uparrow \boldsymbol{V} \uparrow \odot \frac{1}{\Psi} \nabla \left[\frac{1}{G} \nabla \cdot (\boldsymbol{V} \Psi) \right]$$
$$+ \frac{\delta t^2}{24} \left\{ -\frac{8\boldsymbol{V}}{G\Psi} \nabla \cdot \left[\frac{\boldsymbol{V}}{G} \nabla \cdot (\boldsymbol{V}\Psi) \right] + \frac{\partial^2 \boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi} \nabla \cdot \left(\frac{\partial \boldsymbol{V}}{\partial t} \Psi \right) - \frac{2}{G\Psi} \frac{\partial \boldsymbol{V}}{\partial t} \nabla \cdot (\boldsymbol{V}\Psi) \right\}$$

Origin of the term: Upwind differencing with physical velocity

A walk-through of the result

Considered term

$$\overline{\overline{\boldsymbol{V}}} = -\frac{\boldsymbol{\delta x} \odot \boldsymbol{\delta x}}{24} \odot \left[4\boldsymbol{V} \odot \frac{1}{\Psi}\nabla \odot \nabla\Psi + 2\frac{\nabla\Psi}{\Psi} \odot \nabla \odot \boldsymbol{V} + \nabla \odot \nabla \odot \boldsymbol{V}\right]$$
$$+ \frac{\boldsymbol{\delta x}}{2} \odot \left\uparrow\overline{\boldsymbol{V}}\right\uparrow \odot \frac{\nabla\Psi}{\Psi} + \frac{\delta t}{2}\boldsymbol{\delta x} \odot \uparrow\boldsymbol{V}\uparrow \odot \frac{1}{\Psi}\nabla\left[\frac{1}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right]$$
$$+ \frac{\delta t^2}{24}\left\{-\frac{8\boldsymbol{V}}{G\Psi}\nabla\cdot\left[\frac{\boldsymbol{V}}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right] + \frac{\partial^2\boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi}\nabla\cdot\left(\frac{\partial\boldsymbol{V}}{\partial t}\Psi\right) - \frac{2}{G\Psi}\frac{\partial\boldsymbol{V}}{\partial t}\nabla\cdot(\boldsymbol{V}\Psi)\right\}$$

Origin of the term: Upwind differencing with antidiffusive velocity

A walk-through of the result

Considered term

$$
\begin{aligned}
\overline{\overline{\boldsymbol{V}}} = & -\frac{\boldsymbol{\delta x} \odot \boldsymbol{\delta x}}{24} \odot \left[4\boldsymbol{V} \odot \frac{1}{\Psi} \nabla \odot \nabla \Psi + 2 \frac{\nabla \Psi}{\Psi} \odot \nabla \odot \boldsymbol{V} + \nabla \odot \nabla \odot \boldsymbol{V} \right] \\
& + \frac{\boldsymbol{\delta x}}{2} \odot \Big\uparrow \overline{\boldsymbol{V}} \Big\uparrow \odot \frac{\nabla \Psi}{\Psi} + \frac{\delta t}{2} \boldsymbol{\delta x} \odot \uparrow \boldsymbol{V} \uparrow \odot \frac{1}{\Psi} \nabla \left[\frac{1}{G} \nabla \cdot (\boldsymbol{V} \Psi) \right] \\
& + \frac{\delta t^2}{24} \left\{ -\frac{8\boldsymbol{V}}{G\Psi} \nabla \cdot \left[\frac{\boldsymbol{V}}{G} \nabla \cdot (\boldsymbol{V} \Psi) \right] + \frac{\partial^2 \boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi} \nabla \cdot \left(\frac{\partial \boldsymbol{V}}{\partial t} \Psi \right) - \frac{2}{G\Psi} \frac{\partial \boldsymbol{V}}{\partial t} \nabla \cdot (\boldsymbol{V} \Psi) \right\}
\end{aligned}
$$

Origin of the term: Iterative nature of MPDATA

A walk-through of the result

Considered term

$$\overline{\overline{\boldsymbol{V}}} = -\frac{\boldsymbol{\delta x} \odot \boldsymbol{\delta x}}{24} \odot \left[4\boldsymbol{V} \odot \frac{1}{\Psi} \nabla \odot \nabla \Psi + 2 \frac{\nabla \Psi}{\Psi} \odot \nabla \odot \boldsymbol{V} + \nabla \odot \nabla \odot \boldsymbol{V} \right]$$
$$+ \frac{\boldsymbol{\delta x}}{2} \odot \Big\uparrow \overline{\boldsymbol{V}} \Big\uparrow \odot \frac{\nabla \Psi}{\Psi} + \frac{\delta t}{2} \boldsymbol{\delta x} \odot \uparrow \boldsymbol{V} \uparrow \odot \frac{1}{\Psi} \nabla \left[\frac{1}{G} \nabla \cdot (\boldsymbol{V} \Psi) \right]$$
$$+ \frac{\delta t^2}{24} \left\{ -\frac{8\boldsymbol{V}}{G\Psi} \nabla \cdot \left[\frac{\boldsymbol{V}}{G} \nabla \cdot (\boldsymbol{V} \Psi) \right] + \frac{\partial^2 \boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi} \nabla \cdot \left(\frac{\partial \boldsymbol{V}}{\partial t} \Psi \right) - \frac{2}{G\Psi} \frac{\partial \boldsymbol{V}}{\partial t} \nabla \cdot (\boldsymbol{V} \Psi) \right\}$$

Origin of the term: Forward-in-time differencing - transported field

A walk-through of the result

Considered term

$$
\begin{aligned}
\overline{\overline{\boldsymbol{V}}} = & -\frac{\boldsymbol{\delta x}\odot\boldsymbol{\delta x}}{24}\odot\left[4\boldsymbol{V}\odot\frac{1}{\Psi}\nabla\odot\nabla\Psi + 2\frac{\nabla\Psi}{\Psi}\odot\nabla\odot\boldsymbol{V} + \nabla\odot\nabla\odot\boldsymbol{V}\right] \\
& + \frac{\boldsymbol{\delta x}}{2}\odot\left\uparrow\overline{\boldsymbol{V}}\right\uparrow\odot\frac{\nabla\Psi}{\Psi} + \frac{\delta t}{2}\boldsymbol{\delta x}\odot\uparrow\boldsymbol{V}\uparrow\odot\frac{1}{\Psi}\nabla\left[\frac{1}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right] \\
& + \frac{\delta t^2}{24}\left\{-\frac{8\boldsymbol{V}}{G\Psi}\nabla\cdot\left[\frac{\boldsymbol{V}}{G}\nabla\cdot(\boldsymbol{V}\Psi)\right] + \frac{\partial^2\boldsymbol{V}}{\partial t^2} + \frac{2\boldsymbol{V}}{G\Psi}\nabla\cdot\left(\frac{\partial\boldsymbol{V}}{\partial t}\Psi\right) - \frac{2}{G\Psi}\frac{\partial\boldsymbol{V}}{\partial t}\nabla\cdot(\boldsymbol{V}\Psi)\right\}
\end{aligned}
$$

Origin of the term: Forward-in-time differencing - physical velocity

Convergence test in 3D

Test description

- Method of manufactured solutions

$$\Psi(t,\boldsymbol{x}) = (2+\sin t\sin x)(2+\sin t\sin y)(2+\sin t\sin z)$$
$$G(\boldsymbol{x}) = e^{\cos x+\cos y+\cos z}$$
$$V^I(t,\boldsymbol{x}) = \frac{G\cos t}{2+\sin t\sin x^I}$$

  where

$$\boldsymbol{x} = (x,y,z)$$

- Solved on $[0,2\pi]\times[0,2\pi]\times[0,2\pi]$ domain with periodic boundary conditions using N^3 gridpoints

Error norms

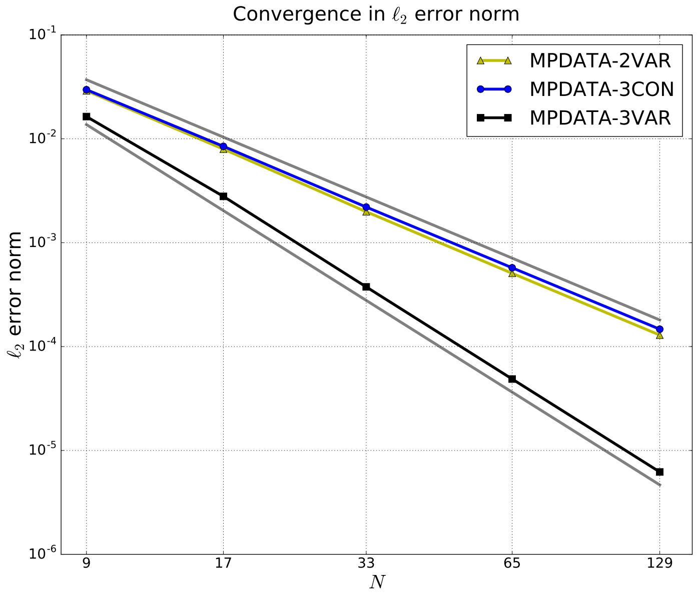
Error norms

$$\ell_2 = \left[\frac{I\left[(\Psi - \Psi_{\text{exact}})^2\right]}{I\left[(\Psi_{\text{exact}})^2\right]} \right]^{1/2}$$

$$\ell_\infty = \frac{\max|\Psi - \Psi_{\text{exact}}|}{\max|\Psi_{\text{exact}}|}$$

$$I(\Psi) = \int \Psi dV$$

Convergence test in 3D - results



Solid-body advection of a cosine bell over the pole

Initial condition

$$\Psi(\lambda,\theta) = \frac{\Psi_0}{4}\left[1+\cos\left(\frac{\pi R_g}{R_c}\right)\right]^2 \text{ if } R_g < R_c \text{ else } 0$$

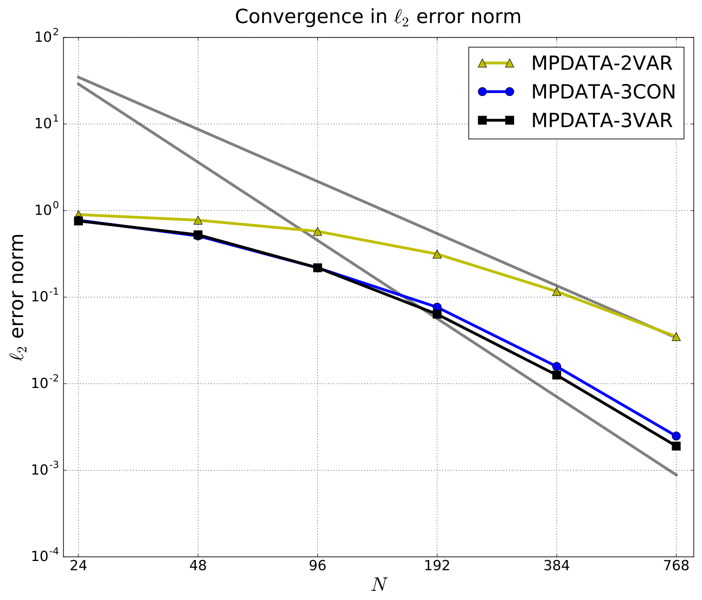
with $\Psi_0 = 1000$ and $R_c = \frac{1}{3}$

$\Psi(\lambda,\theta) \in \mathcal{C}^3$

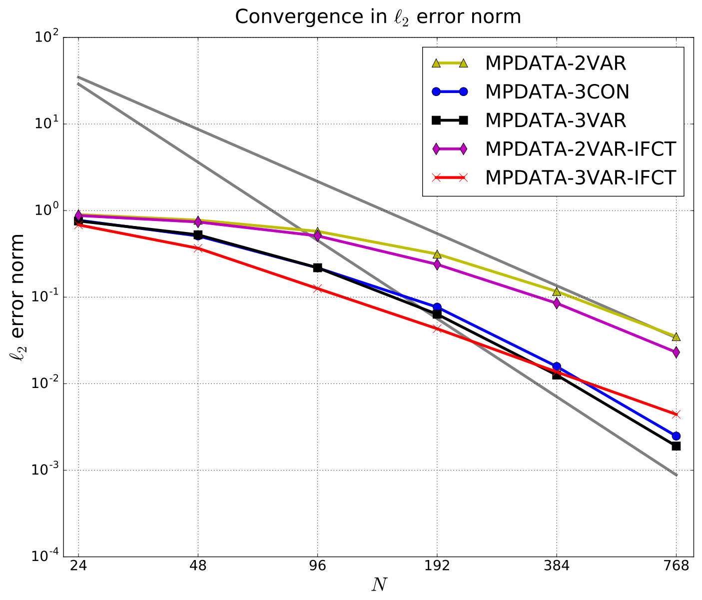
Grid

Regular latitude-longitude grid with $(2N+1) \times N$ points (common to all tests in spherical geometry)

Solid-body advection of a cosine bell over the pole - results



Solid-body advection of a cosine bell over the pole - results with FCT



Stationary vortices on the sphere - Nair and Machenhauer 2002

Wind field

- two vortices located on opposite sides of the sphere rotating with tangential velocity

$$V = v_0 \frac{3\sqrt{3}}{2} \frac{\tanh(\rho)}{\cosh^2(\rho)}$$

$$\rho = \rho_0 \cos(\theta')$$

- location of one of the vortices is chosen to be $(\pi + 0.0025, \pi/2.2)$
- flow field is time-independent

Analytical solution known at any time

$$\Phi(t, \lambda', \theta') = 1 - \tanh\left[\frac{\rho}{\gamma}\sin(\lambda' - \omega_r t)\right]$$

where ω_r is the angular velocity

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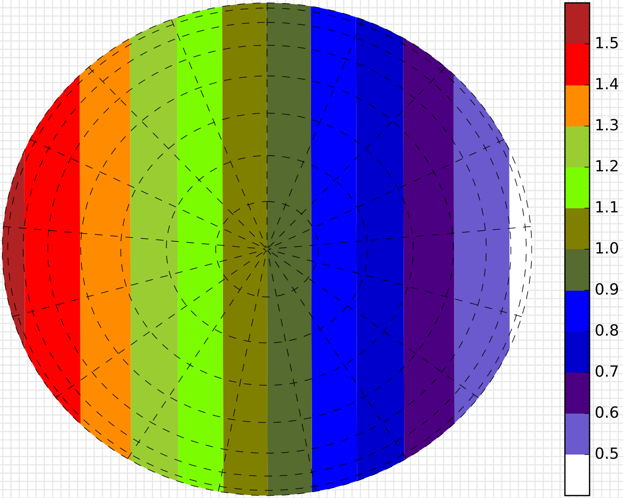

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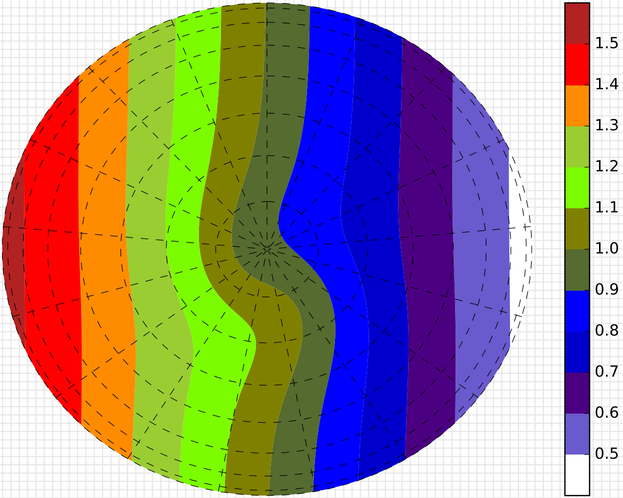

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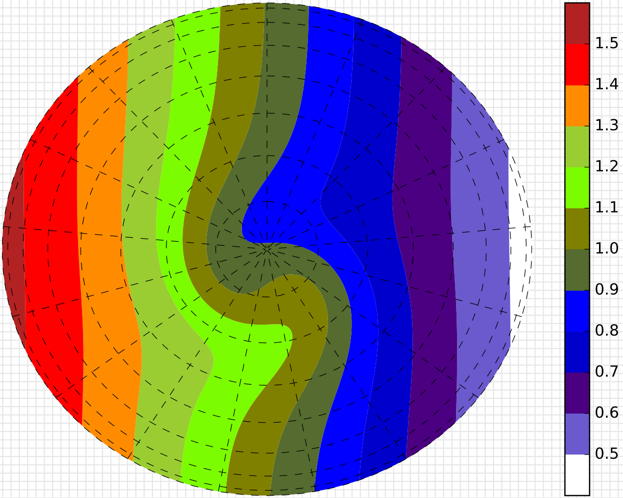

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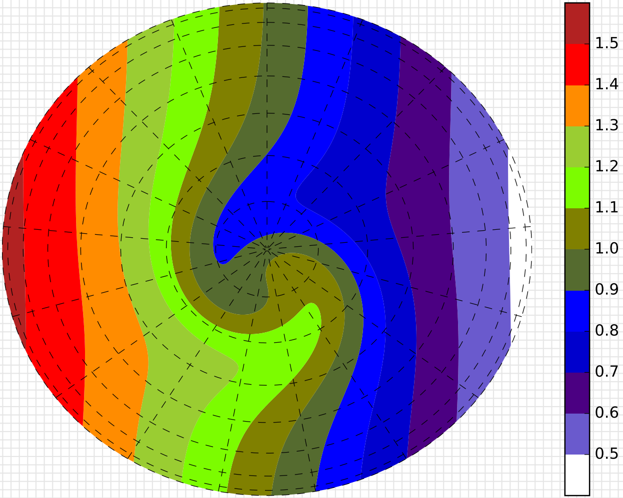

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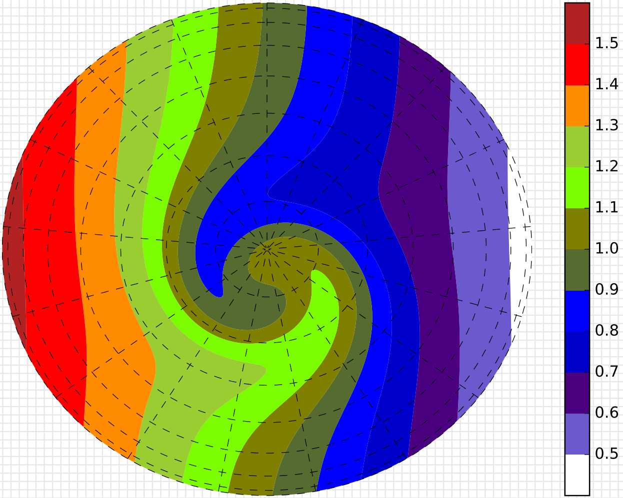

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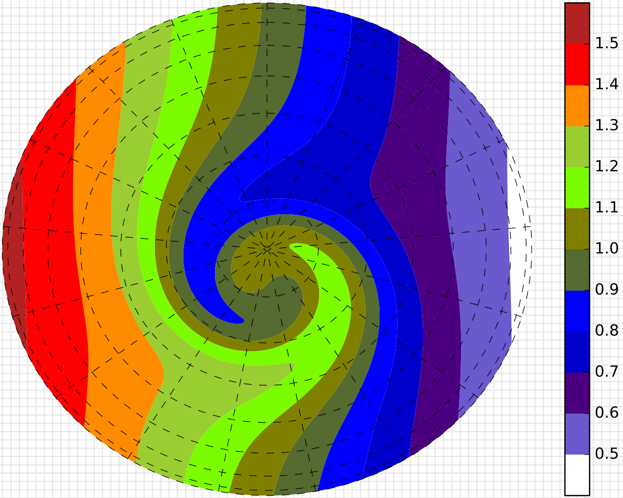

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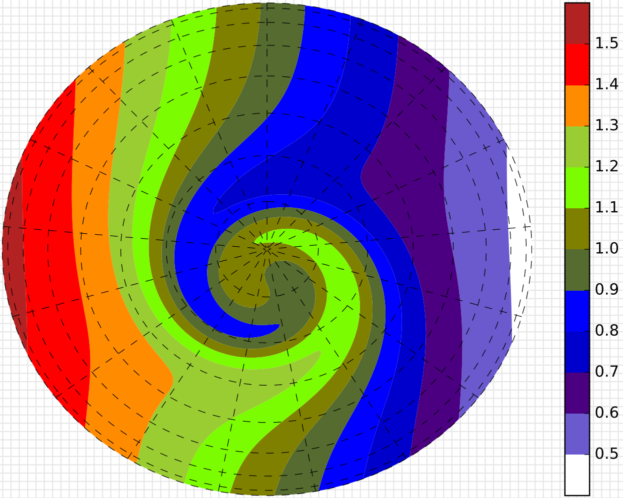

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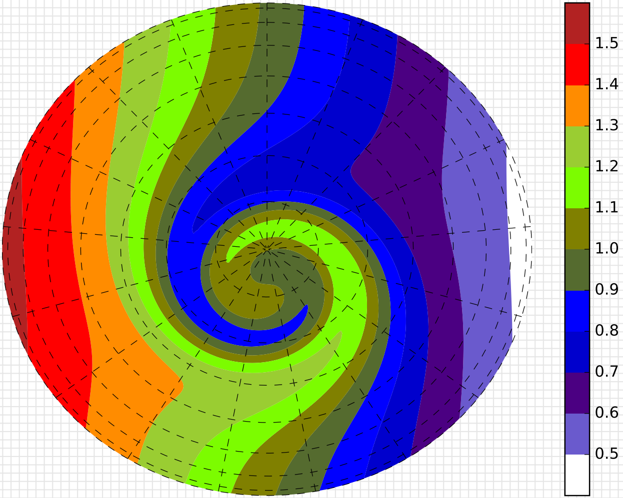
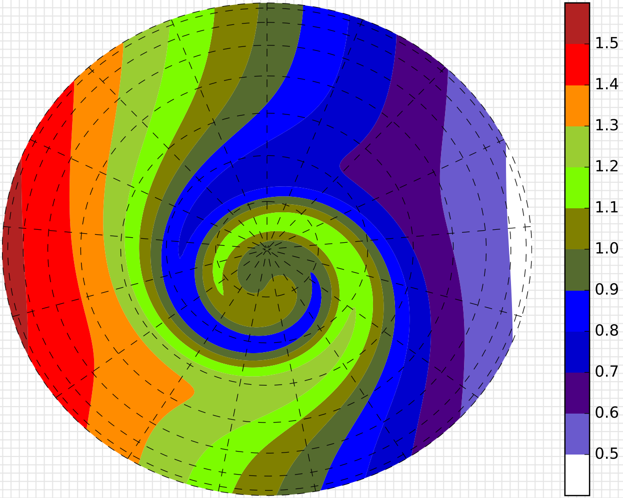


Stationary vortices on the sphere - Nair and Machenhauer 2002



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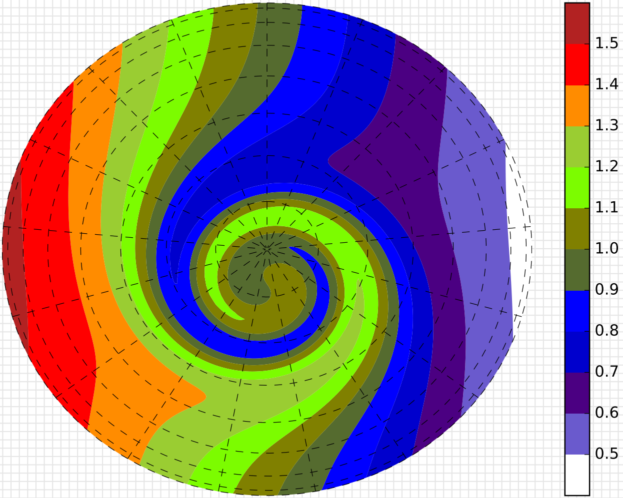

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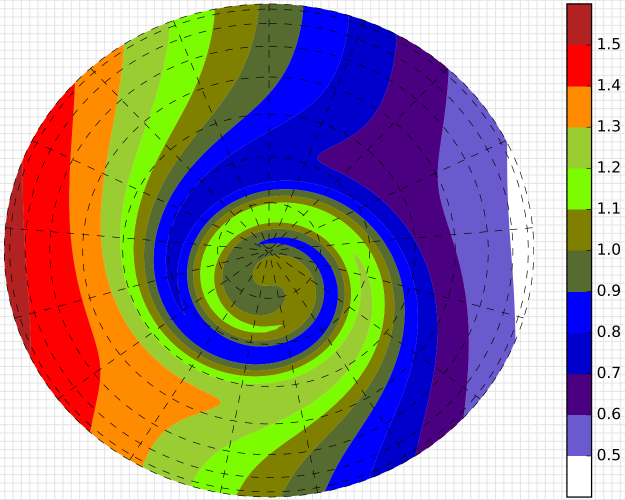

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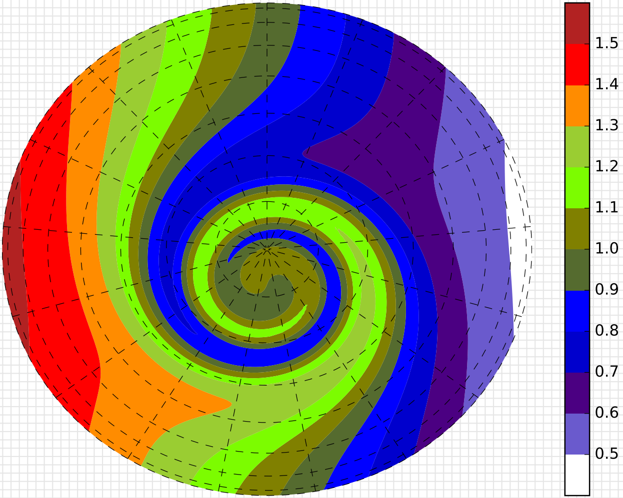

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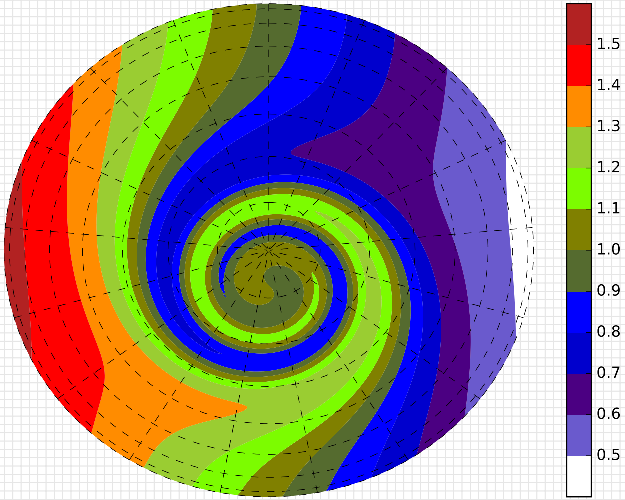
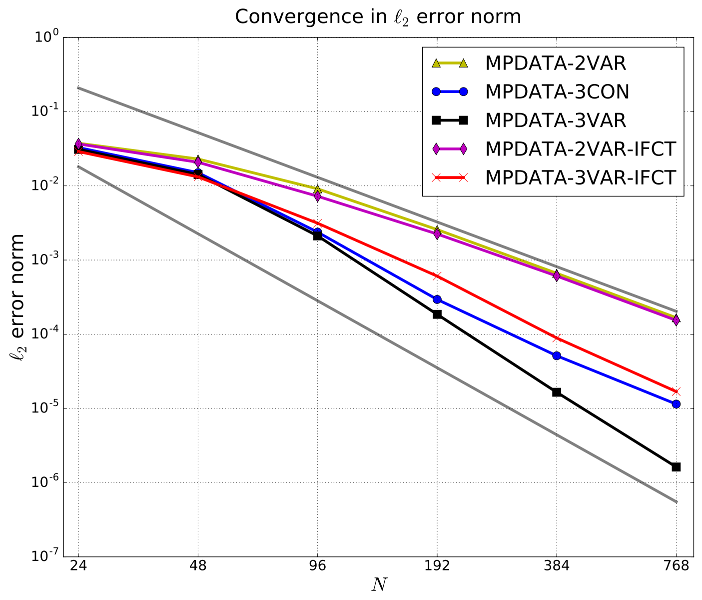
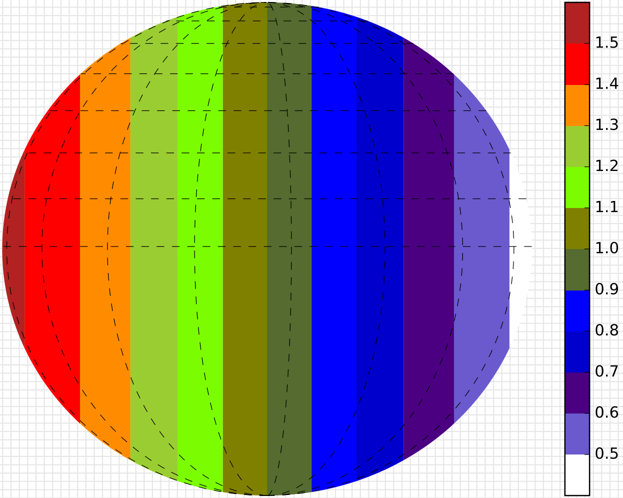


Stationary vortices on the sphere - results



Moving vortices on the sphere - Nair and Jablonowski 2008



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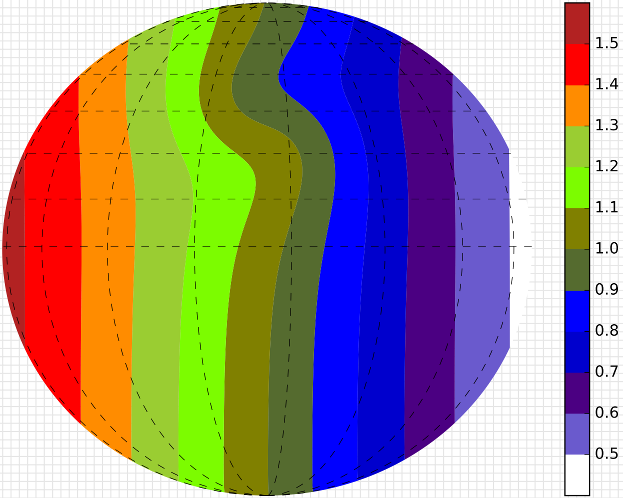
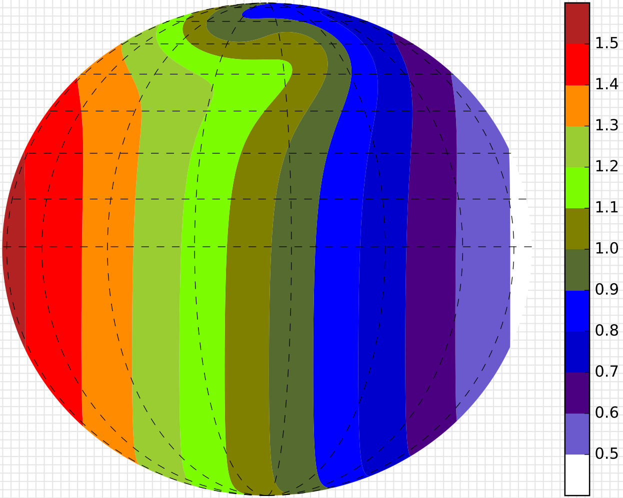


Moving vortices on the sphere - Nair and Jablonowski 2008



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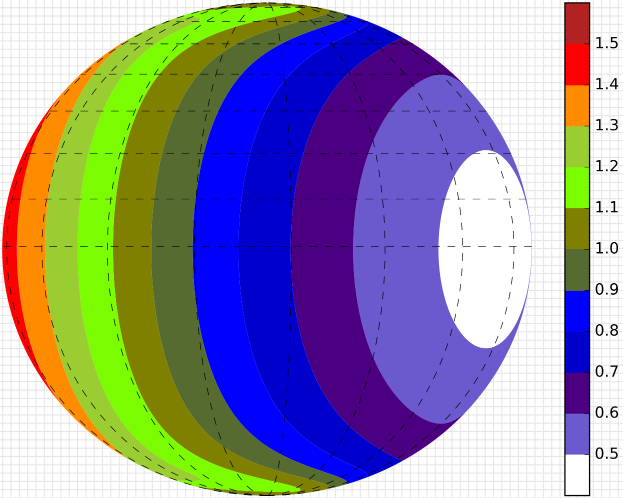
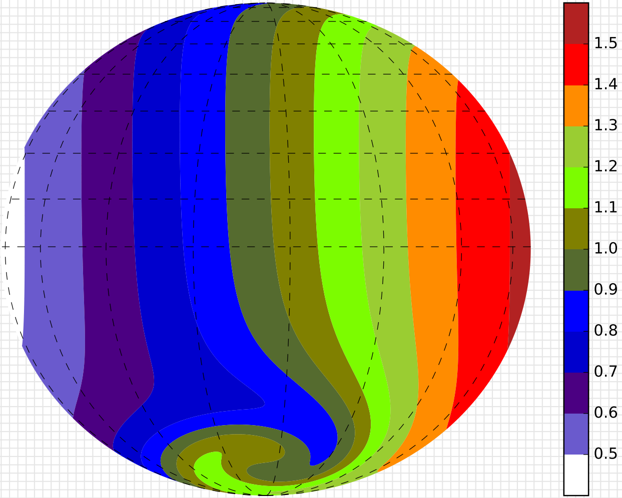
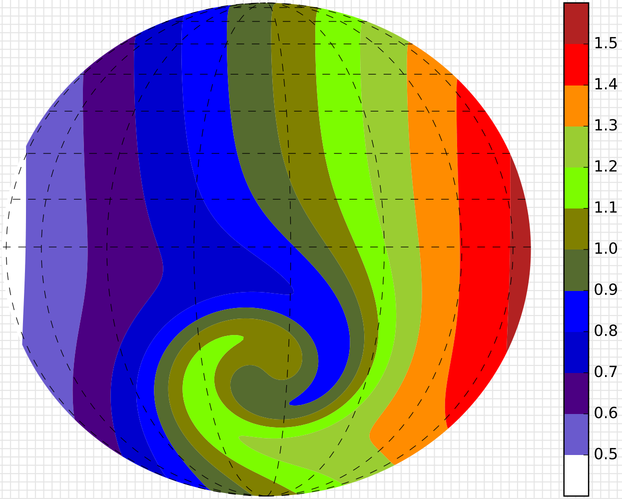


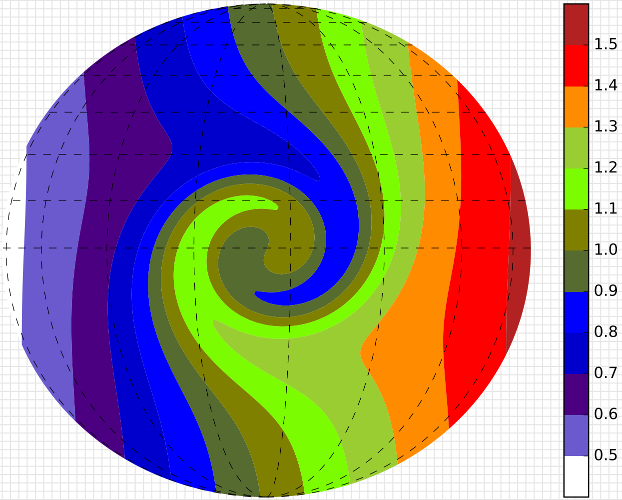
Moving vortices on the sphere - Nair and Jablonowski 2008



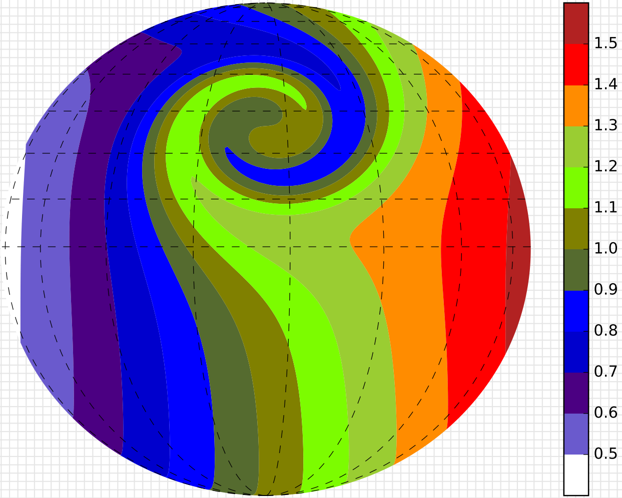
Moving vortices on the sphere - Nair and Jablonowski 2008



Moving vortices on the sphere - Nair and Jablonowski 2008



Moving vortices on the sphere - Nair and Jablonowski 2008



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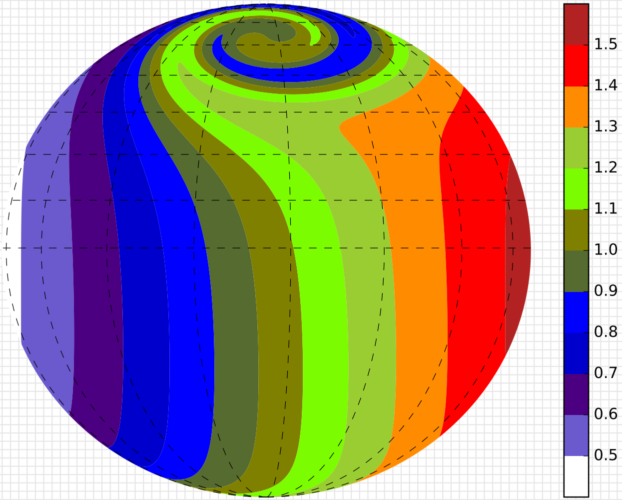
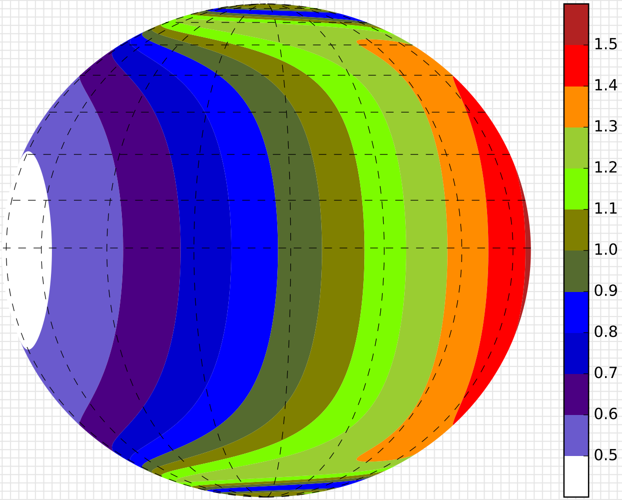
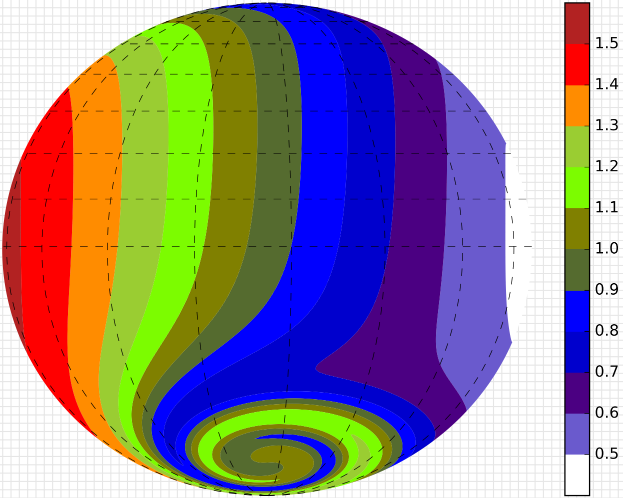


Moving vortices on the sphere - Nair and Jablonowski 2008



Moving vortices on the sphere - Nair and Jablonowski 2008



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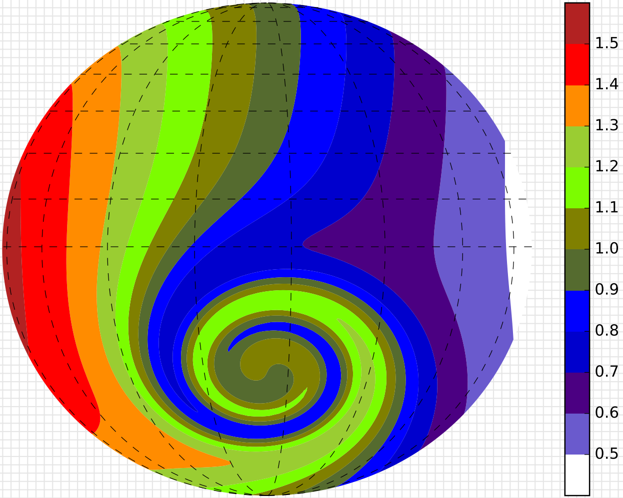
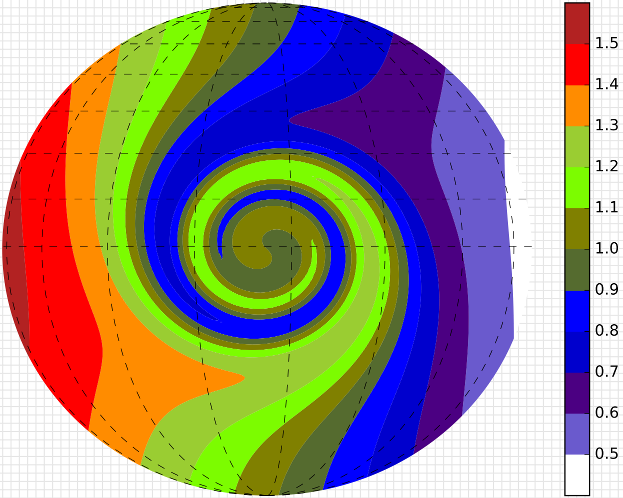
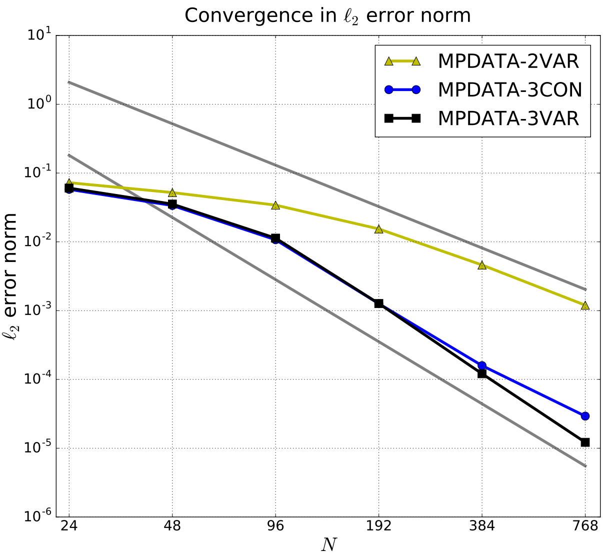


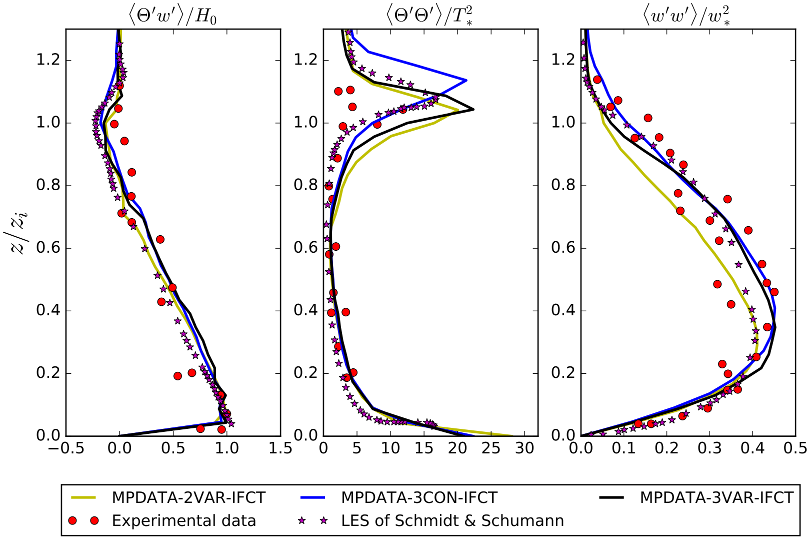
Moving vortices on the sphere - Nair and Jablonowski 2008



Moving vortices on the sphere - results



Beyond pure advection - simulation of convective boundary layer



Third-order accurate MPDATA coming soon to *libmpdata++*

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Repository

`https://github.com/igfuw/libmpdataxx`

Article

Jaruga et al. *libmpdata++ 0.1: a library of parallel MPDATA solvers for systems of generalised transport equations*
Geoscientific Model Development, 8, 1005-1032, 2015

Acknowledgments

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Thank you for your attention !