Stochastic parametrisation models for GFD

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Abstract: Who? Why? How? What?

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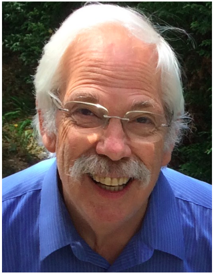
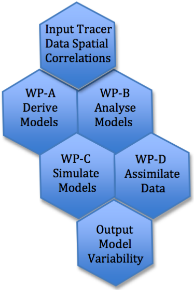
Research Project: Colin Cotter, Dan Crisan, D Holm

Colin Cotter

Dan Crisan



Darryl Holm



Our Project

This project introduces Stochasticity into Partial Differential Equations (SPDEs), Variational Principles (SVPs), Numerical Modelling, Stochastic Data Analysis, and Geophysical Fluid Dynamics (SGFD).

Why? We introduce our methodology as a potential framework for quantifying model transport error.

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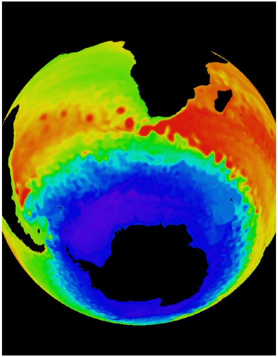
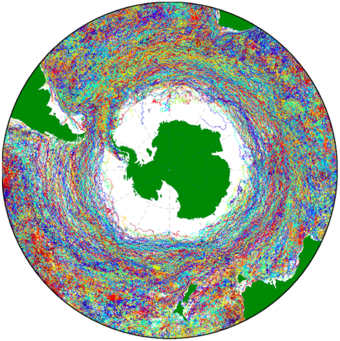
How? to parameterise stochastic transport?

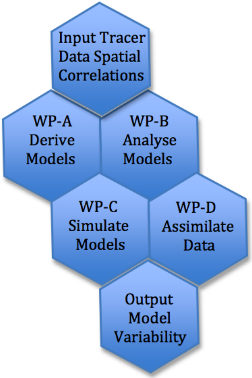
Task: *Learn from stochastic assimilation* of observed data (tracers) how to produce *stochastic fluid motion equations* whose transport parameterisation matches observed statistics / variability of the data.



Numerics

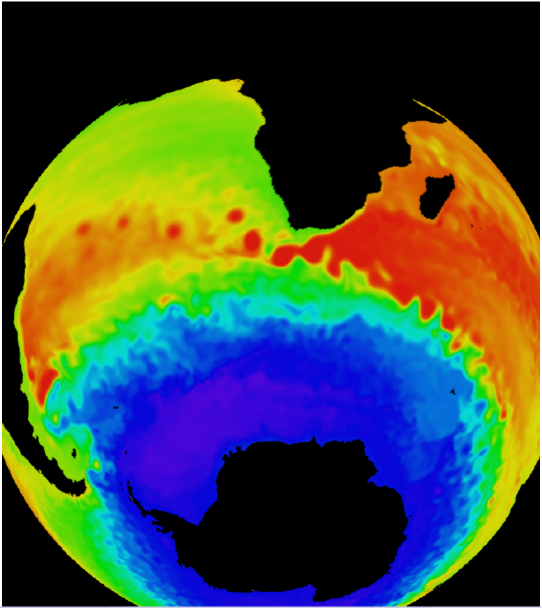


Observations

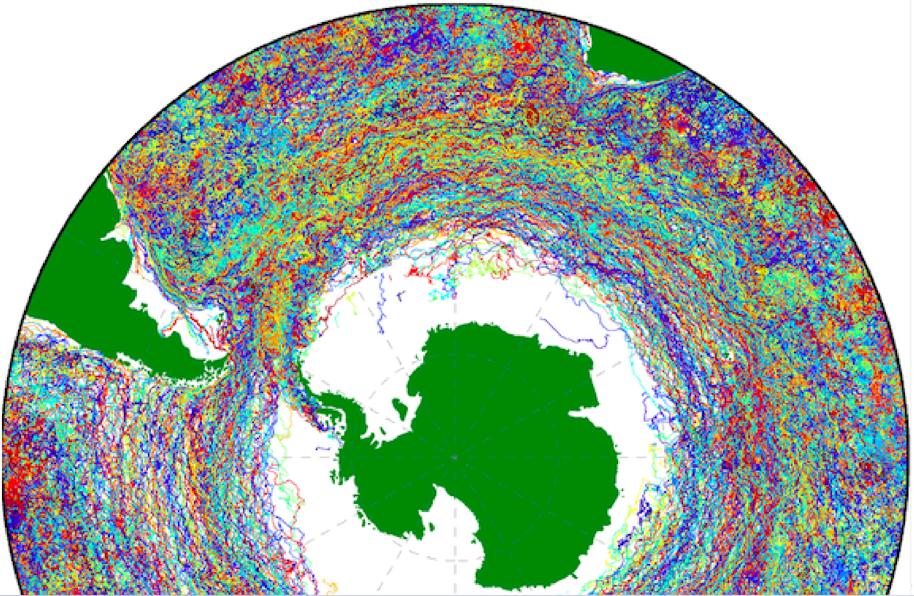


Our Approach

Simulations of sea-surface elevation look like this



Satellite observations look rather like a stochastic flow



How to get the *fluid equations* for these trajectories?

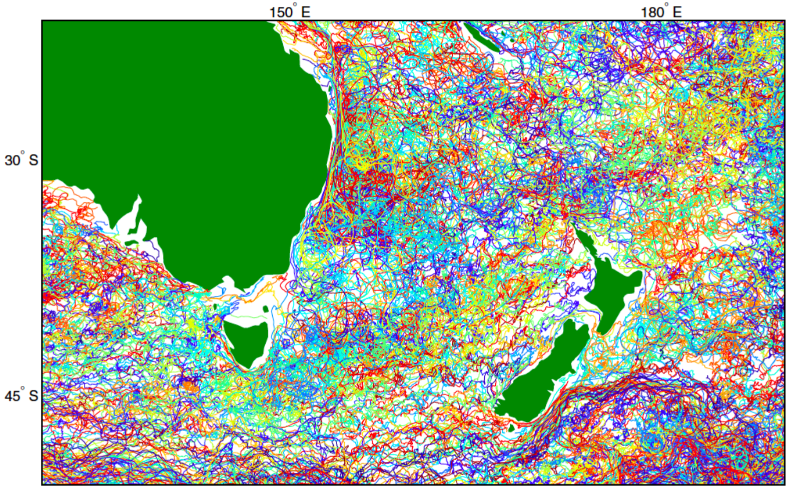


Figure: Here are all surface drifter trajectories since 1980 to have passed between Eastern Australia & New Zealand, courtesy Eric van Sebille [2014].

History: RH Kraichnan [1996, PRL] scalar turbulence

In the Kraichnan model, advection of passive scalar θ is governed by

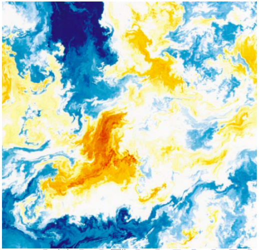
$$d\theta + \underbrace{\mathbf{v}\cdot\nabla\theta}_{\text{Stoch Transport}} = \underbrace{F + \kappa\Delta\theta\, dt}_{\text{Fluct Dissipation}}, \quad \nabla\cdot\mathbf{v} = 0\,,$$

where $\theta(t,\mathbf{r})$ is the scalar (temperature), $F(t,\mathbf{r})$ is the external source, $\mathbf{v}(t,\mathbf{r})$ is the advecting velocity, and κ is diffusivity [Kraichnan(1996)].

Both $F(t,\mathbf{r})$ and $\mathbf{v}(t,\mathbf{r})$ are independent Gaussian *random* functions of t and $\mathbf{r}$, which are δ-correlated in time, e.g., $\mathbf{v}(t,\mathbf{r}) = \sum_k \boldsymbol{\xi}_k(\mathbf{r}) \circ dW_k(t)$.

The $dW_k(t)$ are independent 1D Brownian motions, with $\nabla\cdot\boldsymbol{\xi}_k = 0$ and with bounded trace of the correlation tensor $\sum_k \boldsymbol{\xi}_k \boldsymbol{\xi}_k^T$.

Typical numerical solutions show the *patchiness* in θ associated with intermittency (anomalous scaling).
Very non-Gaussian!



History: R Mikulevicius and BL Rozovskii [MiRo(2005)]

Deriving the stochastic Euler fluid equations

Stochastic paths $x_t = g_t(x_0)$ solve a *Lagrangian* SDE with *prescribed* ξ_t

$$dg_t(x_0) = u_t(g_t(x_0))dt + \xi_t(g_t(x_0)) \circ dW_t\,, \quad \text{with} \quad g_t(x_0) = x_t \in \mathbb{R}^n$$

where $g_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a spatially smooth map depending on time. The corresponding *Eulerian* stochastic *velocity* decomposition is

$$dg_t g_t^{-1} = u_t\, dt + \xi_t \circ dW_t\,, \quad \text{with} \quad g_0(x_0) = x_0 \in \mathbb{R}^n$$

Inserting $dx_t = dg_t(x_0)$ into Newton's 2nd Law [MiRo2004] find SPDE

$$du_t = -[u_t \cdot \nabla u_t + \nabla p - F(u_t)]dt - [\underbrace{\xi_t \cdot \nabla u_t}_{\text{Stochastic Transport}} + \nabla \tilde{p} - G(u_t)] \circ dW_t$$

with $\mathrm{div} u_t = 0$, $\mathrm{div} \xi_t = 0$ and “free forces” $F(u_t)$ and $G(u_t)$.

“Free forces” $F(u_t)$ and $G(u_t)$ regularise serious technical difficulties which arise in taking the 2nd time derivative of g_t in Newton's Law.

Stochastic constrained Hamilton variational principle

The vector field $dx_t = u_t\,dt + \sum_i \xi_i \circ dW_i(t) = dg_t g_t^{-1}$ generates a **Stochastic path**

$$x_t = g_t x_0 = x_0 + \underbrace{\int_0^t u_t(x_t)\,dt}_{\text{Lebesgue}} + \sum_i \underbrace{\int_0^t \xi_i(x_t) \circ dW_i(t)}_{\text{Stratonovich}} .$$

We insert this VF into Hamilton's principle, to constrain the variations:

$$0 = \delta S = \delta \int_0^T \ell(u_t, \underbrace{a_0 g_t^{-1}}_{\text{Advected}})dt + \Big\langle \mu, \circ dg_t g_t^{-1} - u_t\,dt - \sum_i \xi_i \circ dW_i(t) \Big\rangle ,$$

where we vary u, μ and g, with δg=0 at endpoints $[0, T]$.

Definition: Advected quantities $a \in \{b, D \ldots\}$ satisfy $a_t = a_0 g_t^{-1}$, so $da_0 = 0$, along dx_t implies the Eulerian equation $da_t + \mathcal{L}_{dg_t g_t^{-1}} a_t = 0$

$$0 = da_0 = d(a_t g_t) = (da_t + a_t dg_t g_t^{-1})g_t =: (da_t + \mathcal{L}_{dg_t g_t^{-1}} a_t)g_t$$

Deriving SGFD using constrained Hamilton's principle

The stationarity conditions for the stochastic Hamilton's principle are

$$\delta u_t : \quad \frac{\delta \ell}{\delta u_t} = \mu_t \,, \qquad \delta \mu_t : \quad dg_t g_t^{-1} = u\, dt + \sum_i \xi_i(x_t) \circ dW_i(t) = dx_t$$

$$\delta g : \quad \text{Stochastic motion equation,} \qquad d\mu_t + \mathcal{L}_{dg_t g_t^{-1}} \mu_t = \frac{\delta \ell}{\delta a_t} \diamond a_t \, dt \,.$$

Here $a := a_0 g^{-1} \in V^*$ implies $\delta a + \mathcal{L}_{\delta g_t g_t^{-1}} a = 0$ and let's introduce $\delta g_t g_t^{-1} =: \eta \in \mathfrak{X}$ to define the diamond operation $\diamond : V \times V^* \to \mathfrak{X}^*$ as

$$\left\langle \frac{\delta \ell}{\delta a}, \delta a \right\rangle_V = \left\langle \frac{\delta \ell}{\delta a} \,, -\mathcal{L}_\eta a \right\rangle_V =: \left\langle \frac{\delta \ell}{\delta a} \diamond a, \eta \right\rangle_{\mathfrak{X}} \,.$$

The LHS of the motion equation arises by using $d(\delta g) = \delta(dg)$ to prove

$$\delta(dg_t g_t^{-1}) = d\eta - \mathcal{L}_{dg_t g_t^{-1}} \eta \quad \text{in} \quad \left\langle \mu_t \,, \delta(dg_t g_t^{-1}) \right\rangle,$$

then integrating by parts to isolate the coefficient of the VF $\eta = \delta g_t g_t^{-1}$

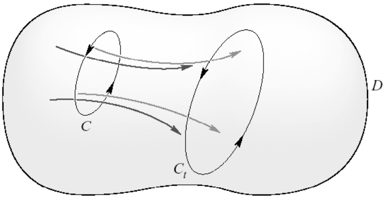
The stochastic Kelvin circulation theorem

The motion equation for this stochastic Hamilton's principle

$$d\mu_t + \mathcal{L}_{dg_t g_t^{-1}} \mu_t = \frac{\delta \ell}{\delta a} \diamond a \, dt \,, \quad \text{with} \quad \frac{\delta \ell}{\delta u_t} = \mu_t \;\; \& \;\; dD_t + \mathcal{L}_{dg_t g_t^{-1}} D_t = 0,$$

implies the **stochastic Kelvin circulation theorem**,

$$d \oint_{c(dg_t g_t^{-1})} \frac{\mu}{D} = \oint_{c(dg_t g_t^{-1})} \underbrace{\left(d\frac{\mu}{D} + \mathcal{L}_{dg_t g_t^{-1}} \frac{\mu}{D} \right)}_{\text{Reynolds transport theorem}} = \oint_{c(dg_t g_t^{-1})} \frac{1}{D} \frac{\delta \ell}{\delta a} \diamond a \, dt$$



- ⋆ Kelvin's thm implies PV is advected by VF, $dx_t = dg_t g_t^{-1}$ (cf. QG).
- ⋆ There are also momentum conservation laws à la [Mémin(2014)].

How did we derive stochastic GFD *motion* equations?

How? Our strategy was to impose stochastic transport of advected quantities [Kraichnan(1996)] as a constraint in Hamilton's principle,

$$0 = \delta S(u,p,a) = \delta \int \Big(\underbrace{\ell(u,a)\, dt}_{\text{Physics}} + \Big\langle p \,,\, \underbrace{da + \mathcal{L}_{dx_t} a}_{\text{Tracer data}} \Big\rangle_V \Big) \,.$$

Here $\ell(u,a)$ is the unperturbed *deterministic* fluid Lagrangian, written as a functional of velocity vector field u, and ...

$\mathcal{L}_{dx_t}$ is the transport operator (Lie derivative) for any advected quantity $a \in V$ by an *Eulerian stochastic vector field*, dx_t,

$$dx_t = dg_t g_t^{-1} = \underbrace{u_t\, dt}_{\text{Drift}} + \sum_k \underbrace{\xi_k \circ dW_k(t)}_{\text{Noise}} \,.$$

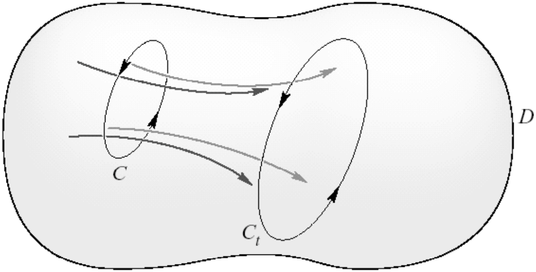
The stochastic vector field dx_t contains *cylindrical Stratonovich noise* whose spatial correlations are given by ξ_k as in [Kraichnan(1996)].

What did we get?

What? *New stochastic GFD models for climate & weather variability.*

New motion equations contain stochastic perturbations which multiply both the solution and its spatial gradient *(in a certain transport way)*.

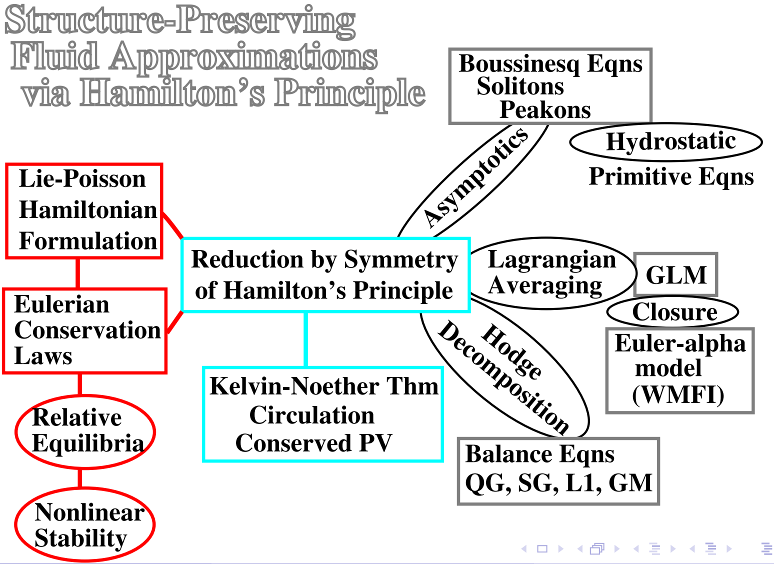
Remarkably, these stochastic GFD models *still preserve* fundamental properties such as Kelvin's circulation theorem and PV conservation.



Examples: *Stochastic QG, RSW, EB, PE, Sound-Proof eqns, etc. [Holm(2015)]*

Eulerian Hamilton's principle & relabelling symmetry!

Hamilton's principle: $\delta S = 0$, with $S = \int \ell(\mathbf{u}, D, b)\, dt$



Conclusion: This is just the geometric framework!

1. The fundamental mathematical structure of fluids is preserved by (1) injecting stochasticity via Hamilton's principle, using (2) a stochastic transport constraint for advected quantities.
2. Deterministic transport becomes stochastic transport.
3. And, stochastic transport still preserves PV (enstrophies).
4. The theory applies to all fluid models derived from Hamilton's principle. (The spatial correlations $\sum_k \xi_k \xi_k^T$ derive from data.)
5. The theory includes stochastic versions of the usual GFD Euler-Boussinesq equations, primitive equations, etc.
6. There's so much more to do, e.g., in analysing and applying these new stochastic GFD equations!
7. Until recently, even the questions of existence and uniqueness for our example of stochastic 2D QG flows were still open!
8. Recently, we have shown long time existence, uniqueness and regularity of 3D stochastic Euler equations derived this way!

Objectives of the new stochastic methodology

- Create new parameterisation approaches in SGFD for mathematics of climate change and weather variability

- Quantify variability in SGFD models due to stochastic transport, by determining the most likely paths of solutions, and their dispersion

- Quantify nonlinear model errors in GFD models by introducing stochastic transport, then determining the most likely paths

- Quantify variability and nonlinear model errors for each member of the new SGFD hierarchy, first for the lowest level approximation, later for higher orders in the GFD asymptotic expansion

- Reduce dimensions by using PV preservation and the dissipative double-bracket operators in the Itô interpretation of these SGFD models as input for finite-horizon parameterising manifolds

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