A Well-Posed and Stable Coupling Procedure for the Compressible and Incompressible Navier-Stokes Equations

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Outlines

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Introduction and motivation

If a well posed problem does not exist:

- An accurate numerical approximation can be made.
- A stable numerical approximation can be made.
- An accurate *and* stable approximation can *not* be made.
- Well-posedness is the *most important* point in coupling procedures.
- Once well-posedness is established, stability follows almost automatically by using the SBP-SAT technique.
- In this talk we focus on well-posedness, and its link to stability.

Coupled problem

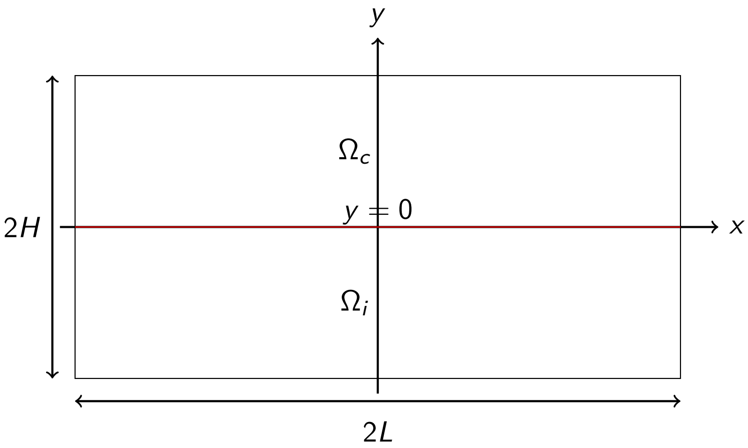

Figure: A schematic of the domains and interface $y = 0$.

The compressible Navier-Stokes equations

The linearized and symmetrized compressible Navier-Stokes equations are

$$U_t + A_1 U_x + A_2 U_y = \epsilon\left(F_x^c + G_y^c\right). \tag{1}$$

The viscous fluxes are given by

$$F^c = A_{11} U_x + A_{12} U_y, \quad G^c = A_{21} U_x + A_{22} U_y,$$

where $U = \left[\frac{\bar{c}\rho}{\sqrt{\gamma}}, \bar{\rho}u, \bar{\rho}v, \frac{\bar{\rho}T}{\bar{c}\sqrt{\gamma(\gamma-1)}}\right]^T.$

The compressible Navier-Stokes equations

The coefficient matrices are:

$$A_1 = \begin{bmatrix} \bar{u} & \frac{\bar{c}}{\sqrt{\gamma}} & 0 & 0 \\ \frac{\bar{c}}{\sqrt{\gamma}} & \bar{u} & 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & 0 & \bar{u} & 0 \\ 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} & 0 & \bar{u} \end{bmatrix}, \quad A_{11} = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda + 2\mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \frac{\gamma\kappa}{\mathrm{Pr}} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \bar{v} & 0 & \frac{\bar{c}}{\sqrt{\gamma}} & 0 \\ 0 & \bar{v} & 0 & 0 \\ \frac{\bar{c}}{\sqrt{\gamma}} & 0 & \bar{v} & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & 0 & \bar{c}\sqrt{\frac{\gamma-1}{\gamma}} & \bar{v} \end{bmatrix} \quad A_{22} = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda + 2\mu & 0 \\ 0 & 0 & 0 & \frac{\gamma\kappa}{\mathrm{Pr}} \end{bmatrix},$$

$$A_{12} = A_{21}^T = \frac{1}{\bar{\rho}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The incompressible Navier-Stokes equations

The linearized incompressible Navier-Stokes equations are:

$$\begin{aligned}
\hat{\rho}(\tilde{u}_x + \tilde{v}_y) =& 0, \\
\hat{\rho}(\tilde{u}_t + \hat{u}\tilde{u}_x + \hat{v}\tilde{u}_y) =& - \tilde{p}_x + \epsilon\hat{\mu}(\tilde{u}_{xx} + \tilde{u}_{yy}), \\
\hat{\rho}(\tilde{v}_t + \hat{u}\tilde{v}_x + \hat{v}\tilde{v}_y) =& - \tilde{p}_y + \epsilon\hat{\mu}(\tilde{v}_{xx} + \tilde{v}_{yy}).
\end{aligned}$$

These equations can be rewritten, using $(\tilde{u}_x + \tilde{v}_y)_x = (\tilde{u}_x + \tilde{v}_y)_y = 0$, as

$$\tilde{I}_3 V_t + B_1 V_x + B_2 V_y = \epsilon(F^i_x + G^i_y), \tag{2}$$

where the viscous fluxes are

$$F^i = B_{11} V_x + B_{12} V_y, \quad G^i = B_{21} V_x + B_{22} V_y,$$

and $\tilde{I}_3 = diag(0, 1, 1)$, $V = \left[\, \tilde{p}, \hat{\rho}\tilde{u}, \hat{\rho}\tilde{v} \,\right]^T$.

The incompressible Navier-Stokes equations

The coefficient matrices are:

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \hat{u} & 0 \\ 0 & 0 & \hat{u} \end{bmatrix}, \qquad B_{11} = \frac{1}{\hat{\rho}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\hat{\mu} & 0 \\ 0 & 0 & \hat{\mu} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \hat{v} & 0 \\ 1 & 0 & \hat{v} \end{bmatrix}, \qquad B_{22} = \frac{1}{\hat{\rho}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{\mu} & 0 \\ 0 & 0 & 2\hat{\mu} \end{bmatrix},$$

$$B_{12} = B_{21}^T = \frac{1}{\hat{\rho}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \hat{\mu} & 0 \end{bmatrix}.$$

The energy method (multiplying the equations by U^T and V^T respectively, and integrating over the spatial domains) leads to

$$\frac{d}{dt}(\|U\|_2^2 + \|V\|_{\tilde{I}_3}^2) + 2\epsilon DI_1 + 2\epsilon DI_2 = -\int_{-L}^{+L} W^T E W|_{y=0} dx,$$

$$\|U\|_2 = \int_{\Omega_c} U^T U d\Omega, \|V\|_{\tilde{I}_3} = \int_{\Omega_i} V^T \tilde{I}_3 V d\Omega$$

. $W = [U, \epsilon G^c, \epsilon G^i, V]^T$

$$E = \begin{bmatrix} -A_2 & \tilde{I}_4 & 0 & 0 \\ \tilde{I}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{I}_3 \\ 0 & 0 & -\tilde{I}_3 & B_2 \end{bmatrix}, \quad \tilde{I}_4 = diag(0, 1, 1, 1).$$

The matrix E has

- 5 five positive eigenvalues
- 4 zero eigenvalues
- five negative eigenvalues

$$\Downarrow$$

5 interface conditions are needed.

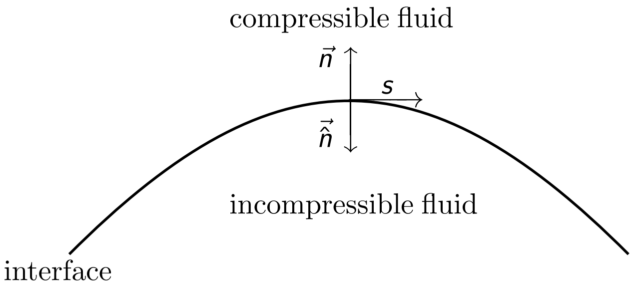

Figure: A sketch of a fluid-fluid interface seprating the two fluids.

Physical intuition

- Mass conservation

$$\bar{\rho}\vec{u}\cdot\vec{n} = -\hat{\rho}\vec{\tilde{u}}\cdot\vec{\hat{n}},$$

- For viscous fluids, in the tangential direction, the no-slip condition holds, i.e,

$$\vec{u}\cdot\vec{s} = \vec{\tilde{u}}\cdot\vec{s},$$

- Conservation of momentum

$$\sigma\vec{n} = -\tilde{\sigma}\vec{\hat{n}}, \quad \sigma = pI_2 - \epsilon\tau, \quad \tilde{\sigma} = \tilde{p}I_2 - \epsilon\tilde{\tau},$$

where

$$\tau = \begin{bmatrix} 2\mu u_x + \lambda(u_x + v_y) & \mu(u_y + v_x) \\ \mu(u_y + v_x) & 2\mu v_y + \lambda(u_x + v_y) \end{bmatrix},$$

$$\tilde{\tau} = \begin{bmatrix} 2\hat{\mu}\tilde{u}_x & \hat{\mu}(\tilde{u}_y + \tilde{v}_x) \\ \hat{\mu}(\tilde{u}_y + \tilde{v}_x) & 2\hat{\mu}\tilde{v}_y \end{bmatrix}.$$

At the interface $y = 0$, we have $\vec{n} = [0,1]^T, \vec{\hat{n}} = [0,-1]^T$ and $\vec{s} = [1,0]^T$ and the interface conditions become

$$
\begin{aligned}
\bar{\rho} v &= \hat{\rho}\tilde{v}, \\
u &= \tilde{u}, \\
p - 2\epsilon\mu v_y - \epsilon\lambda(u_x + v_y) &= \tilde{p} - 2\epsilon\hat{\mu}\tilde{v}_y, \\
\epsilon\mu(u_y + v_x) &= \epsilon\hat{\mu}(\tilde{u}_y + \tilde{v}_x).
\end{aligned} \tag{3}
$$

Well-posedness

We derive the energy rate in the semi-norm

$$\|V\|^2_{\mathcal{H}} = \int_{\Omega_i} V^T \mathcal{H} \tilde{I}_3 V d\Omega, \quad , \quad \mathcal{H} = diag(1, \delta_1, 1).$$

Applying the energy method and inserting the conditions (3) leads to

$$\frac{d}{dt}(\|U\|^2_2 + \|V\|^2_{\mathcal{H}}) + 2\epsilon(DI_1 + \tilde{D}I_2) = + \int_{-L}^{+L} \frac{2\epsilon\bar{\rho}\kappa}{\Pr \bar{c}^2(\gamma - 1)} TT_y\Big|_{y=0} dx$$
$$- \epsilon\mu \int_{-L}^{+L} \Big((\bar{\rho}u - \delta_1\hat{\rho}\tilde{u})(u_x + v_y)\Big)\Big|_{y=0} dx.$$

Well-posedness

The specific choice

$$\delta_1 = \frac{\bar{\rho}}{\hat{\rho}},$$

yields

$$\frac{d}{dt}(\|U\|_2^2 + \|V\|_{\mathcal{H}}^2) + 2\epsilon D I_1 + 2\epsilon D I_2 = \int_{-L}^{+L} \frac{2\epsilon\bar{\rho}\kappa}{\mathrm{Pr}\,\bar{c}^2(\gamma-1)} T T_y \, dx.$$

$\Longrightarrow$ one more condition is needed, as previously indicated.

We add on the decoupled heat equation for the incompressible fluid

$$\hat{\rho}(\tilde{T}_t + \hat{u}\tilde{T}_x + \hat{v}\tilde{T}_y) = \frac{\epsilon\tilde{\kappa}}{\tilde{\mathrm{Pr}}}\Delta\tilde{T}, \quad \tilde{\mathrm{Pr}} = \frac{\mu_\infty \tilde{c}_p}{\kappa_\infty}.$$

By adding the heat equation, six interface conditions are needed.
We use the continuity of temperature and fluxes across the interface

$$T = \tilde{T}, \quad \kappa T_y = \tilde{\kappa}\tilde{T}_y.$$

Updated interface conditions

$$\phi = \tilde{\phi}, \quad \phi = \begin{bmatrix} u \\ p - 2\epsilon\mu v_y - \epsilon\lambda(u_x + v_y) \\ \bar{\rho} v \\ \mu(u_y + v_x) \\ T \\ \kappa T_y \end{bmatrix}, \quad \tilde{\phi} = \begin{bmatrix} \tilde{u} \\ \tilde{p} - 2\epsilon\hat{\mu}\tilde{v}_y \\ \hat{\rho}\tilde{v} \\ \hat{\mu}(\tilde{u}_y + \tilde{v}_x) \\ \tilde{T} \\ \tilde{\kappa}\tilde{T}_y \end{bmatrix}.$$

$$\begin{aligned} \phi =& HU, \quad H = H_0 + H_x \frac{\partial}{\partial x} + H_y \frac{\partial}{\partial y}, \\ \tilde{\phi} =& \tilde{H} V, \quad \tilde{H} = \tilde{H}_0 + \tilde{H}_x \frac{\partial}{\partial x} + \tilde{H}_y \frac{\partial}{\partial y}, \end{aligned}$$

The energy rate will be derived in the new expanded semi-norm

$$\|V\|_{\mathcal{H}}^2 = \int_{\Omega 2} V^T \mathcal{H} \tilde{I}_4 V d\Omega, \qquad \mathcal{H} = diag(1, \delta_1, 1, \delta_2),$$

Applying the energy method and inserting the interface conditions, leads to

$$\frac{d}{dt}(\|U\|_2^2 + \|V\|_{\mathcal{H}}^2) + 2\epsilon D I_1 + 2\epsilon \tilde{D} I_2 = RHS,$$

where

$$RHS = -2\kappa\epsilon \int_{-L}^{+L} T T_y \left(\frac{\bar{\rho}}{\Pr \bar{c}^2(\gamma - 1)} - \frac{\delta_2 \hat{\rho}}{\tilde{\Pr}} \right)\Bigg|_{y=0} dx.$$

The specific choice

$$\delta_2 = \left(\frac{\bar{\rho}}{\hat{\rho}}\right)\left(\frac{\tilde{\Pr}}{\Pr}\right)\frac{1}{\bar{c}^2(\gamma - 1)},$$

leads to $RHS = 0$.

Definition 1

$$
\begin{aligned}
\mathbf{U}_t &= \mathcal{D}_t \mathbf{U} = (D_t \otimes I_4)\mathbf{U} \approx U_t, & D_t &= P_t^{-1} Q_t \otimes I_x \otimes I_y, \\
\mathbf{U}_x &= \mathcal{D}_x \mathbf{U} = (D_x \otimes I_4)\mathbf{U} \approx U_x, & D_x &= I_t \otimes P_x^{-1} Q_x \otimes I_y, \\
\mathbf{U}_y &= \mathcal{D}_y \mathbf{U} = (D_y \otimes I_4)\mathbf{U} \approx U_y, & D_y &= I_t \otimes I_x \otimes P_y^{-1} Q_y,
\end{aligned}
$$

The fully discrete SBP-SAT approximation of problems (1) and (2) are

$$\mathcal{D}_t\mathbf{U}+[D_x\otimes A_1+D_y\otimes A_2]\mathbf{U}-\epsilon(\mathcal{D}_x\mathbf{F}^c+\mathcal{D}_y\mathbf{G}^c)=\mathbb{S}+\mathbb{S}_t,$$
$$\tilde{\mathcal{D}}_t\mathbf{V}+[\tilde{D}_x\otimes B_1+\tilde{D}_y\otimes B_2]\mathbf{V}-\epsilon(\mathcal{D}_x\mathbf{F}^i+\mathcal{D}_y\mathbf{G}^i)=\tilde{\mathbb{S}}+\tilde{\mathbb{S}}_t,$$

$\mathbb{S}$ and $\tilde{\mathbb{S}}$ are given by

$$\mathbb{S}=(I_t\otimes I_x\otimes P_y^{-1}E_0\otimes I_4)\boldsymbol{\Sigma}(\boldsymbol{\phi}_0-\tilde{\boldsymbol{\phi}}_M),$$
$$\tilde{\mathbb{S}}=(I_t\otimes I_x\otimes P_y^{-1}E_M\otimes I_4)\tilde{\boldsymbol{\Sigma}}(\tilde{\boldsymbol{\phi}}_M-\boldsymbol{\phi}_0),$$

where

$$\boldsymbol{\phi}_0=(I_t\otimes I_x\otimes E_0\otimes I_6)\mathbf{H}\mathbf{U},\quad \tilde{\boldsymbol{\phi}}_M=(I_t\otimes I_x\otimes E_M\otimes I_6)\tilde{\mathbf{H}}\mathbf{V}.$$

$\mathbb{S}_t$ and $\tilde{\mathbb{S}}_t$ are given by

$$\mathbb{S}_t=(P_t^{-1}E_0\otimes I_x\otimes I_y\otimes I_4)\boldsymbol{\Sigma}_t^c(\mathbf{U}-\mathbf{f}^c),$$
$$\tilde{\mathbb{S}}_t=(P_t^{-1}E_0\otimes I_x\otimes I_y\otimes I_4)\boldsymbol{\Sigma}_t^i(\mathbf{V}-\mathbf{f}^i),$$

Applying the discrete energy method leads to

$$\|\mathbf{U}_K\|^2_{P_{xy}\otimes I_4} + \|\mathbf{V}_K\|^2_{P_{xy}\otimes\mathcal{H}\tilde{I}_4} + 2\epsilon\mathbf{DI}_1 + 2\epsilon\tilde{\mathbf{D}}\mathbf{I}_2 = \mathbf{IF} + \mathbf{IT},$$

where

$$\begin{aligned}
\mathbf{IF} &= \mathbf{U}^T(P_t\otimes P_x\otimes E_0\otimes A_2)\mathbf{U} - 2\epsilon\mathbf{U}^T(P_t\otimes P_x\otimes E_0\otimes I_4)\mathbf{G}^c \\
&+\mathbf{U}^T(P_t\otimes P_x\otimes E_0\otimes I_4)\mathbf{\Sigma}(\phi_0-\tilde{\phi}_M) \\
&+\left(\mathbf{U}^T(P_t\otimes P_x\otimes E_0\otimes I_4)\mathbf{\Sigma}(\phi_0-\tilde{\phi}_M)\right)^T \\
&-\mathbf{V}^T(P_t\otimes P_x\otimes E_M\otimes\mathcal{H}B_2)\mathbf{V} + 2\epsilon\mathbf{V}^T(P_t\otimes P_x\otimes E_M\otimes\mathcal{H})\mathbf{G}^i \\
&+\mathbf{V}^T(P_t\otimes P_x\otimes E_M\otimes\mathcal{H})\tilde{\mathbf{\Sigma}}(\tilde{\phi}_M-\phi_0) \\
&+\left(\mathbf{V}^T(P_t\otimes P_x\otimes E_M\otimes\mathcal{H})\tilde{\mathbf{\Sigma}}(\tilde{\phi}_M-\phi_0)\right)^T.
\end{aligned} \tag{4}$$

Stability

Proposition 1

By choosing the penalty matrices as

$$\mathbf{\Sigma} = \mathbf{H}^T(I \otimes (A - \frac{A^+}{2})), \tilde{\mathbf{\Sigma}} = -(I \otimes \mathcal{H}^{-1})\tilde{\mathbf{H}}^T(I \otimes \frac{A^+}{2}),$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \bar{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{\epsilon\bar{\rho}}{\Pr c^2(\gamma-1)} \\ 0 & 0 & 0 & 0 & \dfrac{\epsilon\bar{\rho}}{\Pr c^2(\gamma-1)} & 0 \end{bmatrix} = A^+ + A^-,$$

the interface term will be non-positive, i.e. $\mathbf{IF} \leq 0$.

Proposition 2

Choosing the penalty matrices as

$$\Sigma_t^c = -I_4, \quad \Sigma_t^i = -\tilde{I}_4,$$

yields

$$\begin{aligned}\mathbf{IT} = & \|\mathbf{f}_0^c\|^2_{P_{xy}\otimes I_4} + \|\mathbf{f}_0^i\|^2_{P_{xy}\otimes \mathcal{H}\tilde{I}_4} - \|\mathbf{U}_0 - \mathbf{f}_0^c\|^2_{P_{xy}\otimes I_4} - \|\mathbf{V}_0 - \mathbf{f}_0^i\|^2_{P_{xy}\otimes \mathcal{H}\tilde{I}_4} \\ \leq & \|\mathbf{f}_0^c\|^2_{P_{xy}\otimes I_4} + \|\mathbf{f}_0^i\|^2_{P_{xy}\otimes \mathcal{H}\tilde{I}_4}\end{aligned}$$

Proposition (1) and Proposition (2) $\Rightarrow$ Stability.

Numerical results

The rates of convergence are computed as

$$q = \frac{\log(\frac{E_j}{E_{j+1}})}{\log(\frac{N_{j+1}}{N_j})},$$

where E_j is the norm of the error between the approximated and exact solution. N_j denotes the number of grid points at level j.

Numerical results

$N = M$	20	30	40	50
ρ	6.389	3.508	4.494	4.794
$\bar{\rho}u$	5.657	2.910	3.856	4.160
$\bar{\rho}v$	6.101	3.047	3.977	4.271
T	4.886	2.955	3.748	4.061
$\tilde{p}$	5.782	3.654	4.539	4.900
$\hat{\rho}\tilde{u}$	5.432	3.108	3.852	4.160
$\hat{\rho}\tilde{v}$	5.746	3.246	3.933	4.192
$\tilde{T}$	5.515	3.186	3.915	4.211

Table: Convergence rates at $t = 1$, SBP(6,3) in space, SBP(8,4) in time.

Numerical results

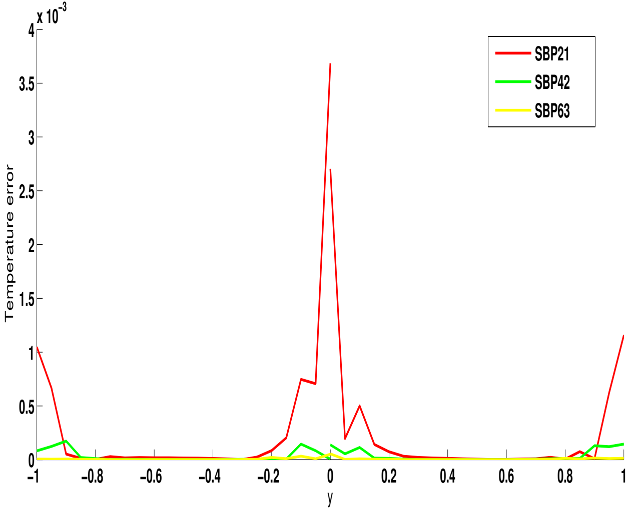

Figure: Temperature error at $x = \frac{1}{2}$ and $N = M = 20$.

Numerical results

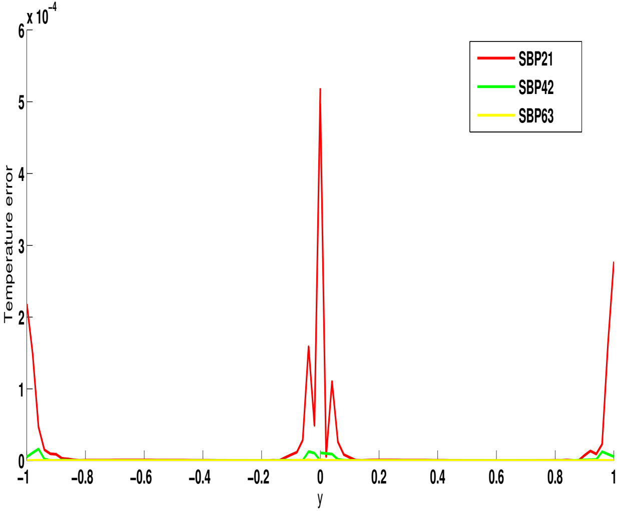

Figure: Temperature error at $x = \frac{1}{2}$ and $N = M = 50$.

Numerical results

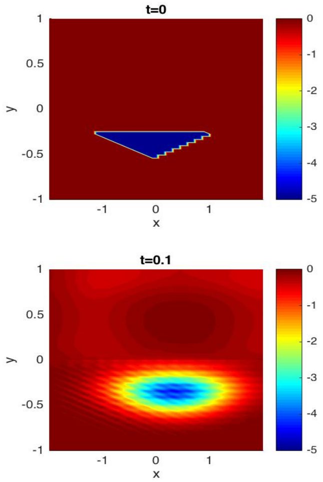

Figure: A sequence of computed temperature with for different times using $M = N = 50$ grid points and third order operators.

Numerical results

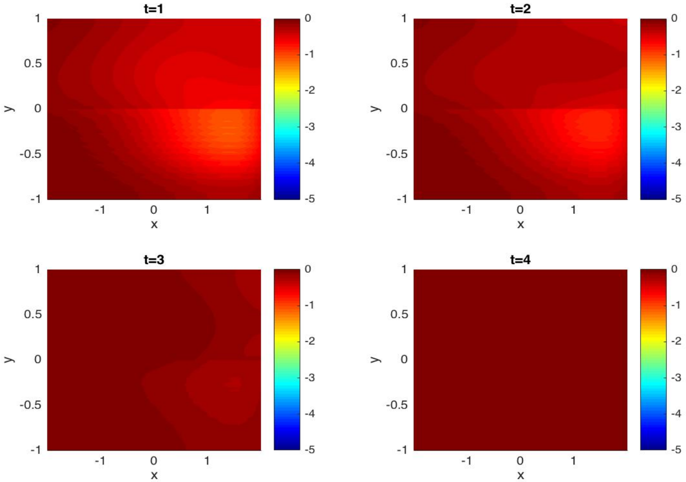

Figure: A sequence of computed temperature with for different times using $M = N = 50$ grid points and third order operators.

Summary and conclusions

- We have discussed the coupling of compressible and incompressible Navier-Stokes equations
- The decoupled heat equation was added to the incompressible equations in order to obtain a sufficient number of interface conditions
- It was shown that the coupled problem with the physical interface conditions satisfy an energy estimate
- Stability and accuracy followed immediately form the well-posedness results using SBP-SAT technique
- The convergence rates were verified by the method of manufactured solutions and the results were consistent with the theory within the SBP framework

Thank you for listening!