Coupling Convection with the Continuity Equation – a Multi-fluid approach

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Motivation

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- Here is how to do it

Conditional Filtering [Thuburn et al., 2018]

- Each location of the continuous fluid is given one of a number of labels, depending on the model complexity. Eg:

$$I_0(\mathbf{x},t) = \begin{cases} 1 & \text{if fluid is in stable atmosphere} \\ 0 & \text{otherwise} \end{cases}$$

$$I_1(\mathbf{x},t) = \begin{cases} 1 & \text{if fluid is in a buoyant plume} \\ 0 & \text{otherwise} \end{cases}$$

$$I_2(\mathbf{x},t) = \begin{cases} 1 & \text{if fluid is in a downdraft} \\ 0 & \text{otherwise} \end{cases}$$

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- Apply a spatial filter, for example averaging over grid boxes:

$$\begin{aligned} \sigma_i &= \overline{I_i} \\ \rho_i &= \overline{I_i \rho} \\ \rho_i \mathbf{u}_i &= \overline{I_i \rho \mathbf{u}} \\ \rho_i \theta_i &= \overline{I_i \rho \theta} \end{aligned}$$

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- Derive equations of motion for each fluid and parametrise interactions

Conditionally Filter the Compressible Euler Equations

$$\frac{\partial \sigma_i \rho_i \mathbf{u}_i}{\partial t} + \nabla \cdot \left(\sigma_i \rho_i \mathbf{u}_i \mathbf{u}_i + \mathsf{F}_{\mathrm{SF}}^{\mathbf{u}_i} \right) + c_p \overline{I_i \rho \theta \nabla \pi} - \sigma_i \rho_i \mathbf{g} = \sum_{j \neq i} \left(\sigma_j \rho_j \mathbf{u}_j S_{ji} - \sigma_i \rho_i \mathbf{u}_i S_{ij} \right)$$

$$\frac{\partial \sigma_i \rho_i}{\partial t} + \nabla \cdot \left(\sigma_i \rho_i \mathbf{u}_i \right) = \sum_{j \neq i} \left(\sigma_j \rho_j S_{ji} - \sigma_i \rho_i S_{ij} \right)$$

$$\frac{\partial \sigma_i \rho_i \theta_i}{\partial t} + \nabla \cdot \left(\sigma_i \rho_i \mathbf{u}_i \theta_i + \mathbf{F}_{\mathrm{SF}}^{\theta_i} \right) = \sum_{j \neq i} \left(\sigma_j \rho_j \theta_j S_{ji} - \sigma_i \rho_i \theta_i S_{ij} \right)$$

Note sub-filter scale fluxes (due to non-linearities):

$$\begin{aligned} \overline{I_i \rho \mathbf{u} \theta} &= \sigma_i \rho_i \mathbf{u}_i \theta_i + \mathbf{F}_{\mathrm{SF}}^{\theta_i} \\ \overline{I_i \rho \mathbf{u} \mathbf{u} \theta} &= \sigma_i \rho_i \mathbf{u}_i \mathbf{u}_i + \mathsf{F}_{\mathrm{SF}}^{\mathbf{u}_i} \end{aligned}$$

Assume that pressure is uniform between fluids and that:

$$\overline{I_i \rho \theta \nabla \pi} = \sigma_i \rho_i \theta_i \nabla \pi + \sum_{j \neq i} \sigma_i \sigma_j \mathbf{d}_{ij} + \mathbf{F}_{\mathrm{SF}}^{\Pi_i}$$

- d_{ij} is drag exerted by fluid j on fluid i
- $\sigma_i \rho_i S_{ij}$ is mass transfer rate from fluid i to fluid j
- Straightforward to do the same for moisture variables

Advective Form

Need to solve in advective (or vector invariant) form to avoid problems when $\sigma_i \to 0$ and for bounded advection of σ_i. Ignoring sub-filter-scale fluxes:

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -2\Omega \times \mathbf{u}_i - c_p \theta_i \nabla \pi + \mathbf{g} + \sum_{j \neq i} \left(\frac{\sigma_j \rho_j}{\sigma_i \rho_i} S_{ji} (\mathbf{u}_j - \mathbf{u}_i) - \mathbf{D}_{ij} \right)$$

$$\frac{\partial \sigma_i \rho_i}{\partial t} + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = \sum_{j \neq i} (\sigma_j \rho_j S_{ji} - \sigma_i \rho_i S_{ij})$$

$$\frac{\partial \theta_i}{\partial t} + \mathbf{u}_i \cdot \nabla \theta_i = \sum_{j \neq i} \left(\frac{\sigma_j \rho_j}{\sigma_i \rho_i} S_{ji} (\theta_j - \theta_i) \right)$$

Equation of State $p_0 \pi^{\frac{1-\kappa}{\kappa}} = R \rho_i \theta_i = R \rho \theta = R \sum_i \sigma_i \rho_i \theta_i$

Numerical Solution [Weller and McIntyre, submitted]

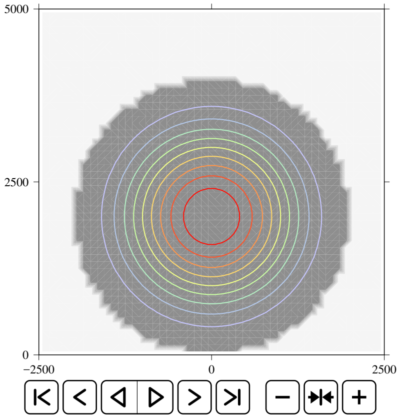
- Finite Volume Advection
  - Bounded advection of $\sigma_i \rho_i$ (TVD scheme with van-Leer limiter)
  - θ_i and $\mathbf{u}_i$ advected using using flux form operators:

$$\mathbf{u}_i \cdot \nabla \theta_i = \nabla \cdot (\theta_i \mathbf{u}_i) - \theta_i \nabla \cdot \mathbf{u}_i$$

- Lorenz C-grid staggering
- Semi-implict: implicit acoustic waves
  - Velocity and density in each fluid expressed as a function of Exner pressure, π
  - Substituted into continuity equation to get Helmholtz equation for π

Rising Bubble, two fluids with different initial conditions

- No transfer terms, stabilisation or sub-filter fluxes
- Two initially hydrostatically balanced, stationary fluids
- Fluid 0 (stable fluid):
  - $\theta_0 = 300\text{K}$
  - $\sigma_0 = \begin{cases} 0.5 & \text{circle near the ground} \\ 1 & \text{elsewhere} \end{cases}$
- Fluid 1 (buoyant fluid):
  - $\theta_1 = \begin{cases} 300\text{K} + \theta' & \text{in circle} \\ 300\text{K} & \text{elsewhere} \end{cases}$
  - $\sigma_1 = \begin{cases} 0.5 & \text{circle near the ground} \\ 0 & \text{elsewhere} \end{cases}$



Stabilisation

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- Diffusion between fluids (diffuse σ_i) Weller and McIntyre [submitted]

$$\sigma_i \rho_i S_{ij} = \frac{K_\sigma}{2} \max\left(\nabla^2 \left(\sigma_j \rho_j - \sigma_i \rho_i\right),\ 0\right)$$

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- Drag between fluids

$$\mathbf{D}_{ij} = \frac{\sigma_j}{\rho_i} \frac{C_D \overline{\rho}}{r_c} \left|\mathbf{u}_i - \mathbf{u}_j\right| \left(\mathbf{u}_i - \mathbf{u}_j\right)$$

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- Remove divergence local to just one fluid: Weller and McIntyre [submitted]

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- Diffusion of vertical velocity (a sub-filter-scale flux) (John Thuburn)

Stabilisation by removing divergence local to one fluid

Continuity equation:

$$\frac{\partial \sigma_i \rho_i}{\partial t} + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = \sum_{j \neq i} (\sigma_j \rho_j S_{ji} - \sigma_i \rho_i S_{ij})$$

Transfer converging fluid:

$$\sigma_i \rho_i S_{ij} = \frac{1}{2} \max(\sigma_j \rho_j \nabla \cdot \mathbf{u}_j - \sigma_i \rho_i \nabla \cdot \mathbf{u}_i,\ 0)$$

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$$\frac{\partial \sigma_i \rho_i}{\partial t} + \mathbf{u}_i \cdot \nabla(\sigma_i \rho_i) = -\frac{1}{2} \overline{\rho \nabla \cdot \mathbf{u}}$$

Stable treatment of transfer terms

The transfer terms can be large $\therefore$

- Operator splitting to ensure boundedness
- Implicit treatment for stability ($\frac{\sigma_j \rho_j}{\sigma_i \rho_i} S_{ji} \to \infty$ as $\sigma_i \rho_i \to 0$)

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For example for θ_i:

$$\text{Advection:} \qquad \theta_i' = \theta_i^n - \Delta t\left\{(1-\alpha)\mathbf{u}_i^n \cdot \nabla\theta_i^n + \alpha\mathbf{u}_i' \cdot \nabla\theta_i'\right\}$$

$$\text{Transfers:} \qquad \theta_i^{n+1} = \theta_i' + \Delta t \sum_{j\neq i} \frac{\sigma_j\rho_j}{\sigma_i\rho_i} S_{ji}\left(\theta_i^{n+1} - \theta_j^{n+1}\right)$$

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Shorthand: $T_{ij} = \Delta t \frac{\sigma_j \rho_j}{\sigma_i \rho_i} S_{ji}$ and re-arrange for $i = 0, 1, 2$:

$$\begin{pmatrix} 1 + T_{01} + T_{02} & -T_{01} & -T_{02} \\ -T_{10} & 1 + T_{10} + T_{12} & -T_{12} \\ -T_{20} & -T_{21} & 1 + T_{21} + T_{21} \end{pmatrix} \begin{pmatrix} \theta_0^{n+1} \\ \theta_1^{n+1} \\ \theta_2^{n+1} \end{pmatrix} = \begin{pmatrix} \theta_0' \\ \theta_1' \\ \theta_2' \end{pmatrix}$$

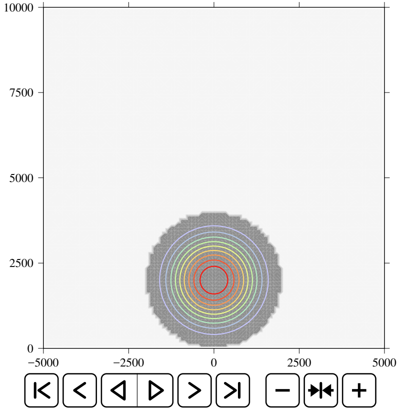
Drag Between Fluids

From formula for drag on a rising bubble

$$D_{ij} = \sigma_j \frac{C_D}{r_c} \frac{\bar{\rho}}{\rho_i} \left| \mathbf{u}_i - \mathbf{u}_j \right| \left(\mathbf{u}_i - \mathbf{u}_j \right)$$

r_c = bubble or plume radius.
As σ_i becomes small we need r_c to become small so that the drag is large and the vanishing fluid moves with the mean flow:

$$r_c = \max \left(r_{\min},\ r_{\max} \prod_i \sigma_i \right) \quad (1)$$



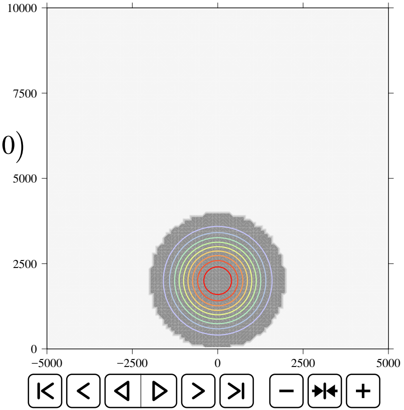
Try $C_D = 1$, $r_{\min} = 100$m, $r_{\max} = 2000$m

Diffusion Between Fluids

- Similar to convective entrainment
- Diffusion coefficient, K_σ, could be chosen based on wind shear

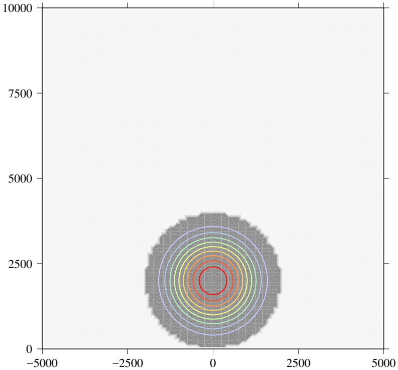
$$\sigma_i \rho_i S_{ij} = \frac{K_\sigma}{2} \max\left(\nabla^2\left(\sigma_j \rho_j - \sigma_i \rho_i\right),\ 0\right)$$

- Total mass is not diffused
- Will control oscillations in σ
- Try $K_\sigma = 200\ \mathrm{m^2 s^{-1}}$



Transfer Converging Fluid

- Removes divergence that is local to one fluid
- Equation for σ_i becomes bounded
- $\sigma_i \rho_i S_{ij} = \frac{1}{2} \max\left(\sigma_j \rho_j \nabla \cdot \mathbf{u}_j - \sigma_i \rho_i \nabla \cdot \mathbf{u}_i,\ 0\right)$
- No arbitrary coefficients

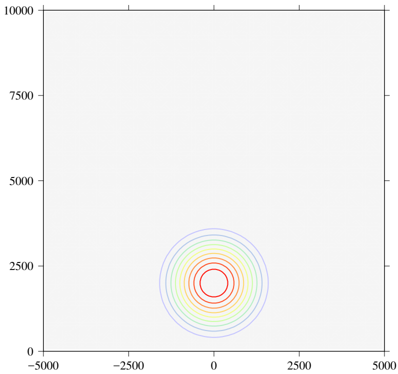


Mass Transfer based on Buoyancy Perturbations for Convection

- Positive θ_0 anomalies should be transferred to fluid one
- Write this in terms of PDEs
- How do we diagnose this without using a reference state?

$$S_{01} = \begin{cases} -K_\theta \frac{\nabla^2 \theta_0}{\theta_0} & \text{when } \nabla^2 \theta_0 < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$S_{10} = \begin{cases} K_\theta \frac{\nabla^2 \theta_1}{\theta_1} & \text{when } \nabla^2 \theta_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$



- Divergence transfer to stabilise
- Simulation using $\sigma_0 = 0$ everywhere initially
- Warm anomaly initially in fluid 0
- $K_\theta = 10^6\ \mathrm{m^2 s^{-1}}$

Conclusions

- Stable numerical method for solving advective form multi-fluid equations
- Forms of stabilisation:
  - Diffusion of mass between fluids
  - Drag between fluids
  - Transfer converging fluid - no parameters to set
- To mimic convective parameterisation, transfer based on $\nabla^2\theta_i$

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